Conjectures on Bridgeland stability for derived Fukaya categories

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Background on symplectic topology

Calabi-Yau Manifolds

Definition

A Calabi-Yau n-fold is a n-dimensional complex Kähler manifold M, such that, if ω is the Kähler form, there is a holomorphic $(n, 0)$-form Ω such that

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega}.$$

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A CY n-fold is symplectic. Let L be an oriented Lagrangian (embedded or immersed - singular!). A *grading* is a smooth function $\theta_L : L \to \mathbb{R}$ (phase) such that $\Omega|_L = e^{i\theta_L} dV_L$, where dV_L is a volume form on L. A *graded Lagrangian* is thus a pair (L, θ_L).

Calabi-Yau Manifolds

Examples:

- $\mathbb{C}^n$ with coordinates $z_1, ..., z_n$ and standard complex structure and metric is an (exact) CY n-fold via:

$$\omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \cdots\cdots \mathrm{d}z_n \wedge \mathrm{d}\overline{z_n}), \quad \Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n.$$

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- The Fermat quintic: $\{[z_0 : \cdots : z_4] \in \mathbb{C}P^4 \,|\, z_0^4 + \cdots + z_4^4 = 0\}$ is a CY 3-fold, with Kahler metric given by restricting the Fubini-Study metric.

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- Consider a complex 2-torus A and consider the orbifold $A/\mathbb{Z}/2\mathbb{Z}$ acting by $x \mapsto -x$. This has 16 points that look like the vertex of a cone. Blowing up these points gives a CY 2-fold, called a Kummer surface.

The derived Fukaya category

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Let M be a CY n-fold. The Fukaya category of M, $\operatorname{Fuk} M$ is the A_∞ category with objects embedded Lagrangian branes with $HF^\bullet$ unobstructed, and morphisms are the ($\mathbb{Z}$-)graded Λ_{nov}-modules $CF^\bullet(L, L')$ with the A_∞ operations we discussed in class.

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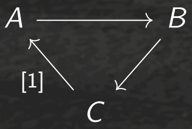

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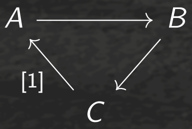

- Triangles must satisfy certain axioms.

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- Given a partial morphism of triangles, there is exists a 'filling' (possibly not unique!)

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- the octahedral axiom. Roughly speaking, given $f : A \to B$ and $g : B \to C$ this compares the triangles associated to f and g and the triangle associated to $g \circ f$.

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Fact: $\mathrm{Kom}(\mathcal{A})$ is triangulated, with shift functor [1] being the cohomological shift of complexes, and distinguished triangles are all of the form

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One can go further and "invert" all quasi-isomorphisms (maps which induce isomorphisms on cohomology) in $\mathrm{Kom}(\mathcal{A})$. The resulting category is called the derived category $\mathrm{D}(\mathcal{A})$, which is also triangulated.

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Now if X is a projective variety, then coherent sheaves are the modules over $\mathcal{O}_X$ which are glued together from the affine case above. That is, F is coherent if there is an affine open cover $X = \cup U_i$, $U_i = \operatorname{Spec} A_i$ such that $F|_{U_i}$ can be identified with a finitely generated module over A_i. The category $\operatorname{coh} X$ is abelian.

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Define the category $\mathrm{D}^b(X) = \mathrm{D}^b(\operatorname{coh} X)$ to be the subcategory of $\mathrm{D}(\operatorname{coh} X)$ consisting of complexes which are concentrated in degrees lying in a bounded interval $I \subset \mathbb{Z}$.

The derived Fukaya category

Fuk M does not fit into the previous example, so it requires special care. Let $\mathcal{A}$ be an A_∞ category.

Step 1. Additive enlargement: Objects become formal direct sums of formal shifts. That is, define $\Sigma\mathcal{A}$ to be the category with objects finite sums $\oplus X_i[k_i]$ with $X_i \in \mathcal{A}$ and $k_i \in \mathbb{Z}$.

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Step 2. Twisted complexes: Define the category $\mathrm{Tw}\,\mathcal{A}$ whose objects are pairs (X, δ_X) with $X \in \Sigma\mathcal{A}$ and $\delta_X = (\delta_X^{ij}) \in \mathrm{Hom}_{\Sigma\mathcal{A}}(X, X)$ is a lower triangular matrix such that

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- $\mathrm{Tw}\,\mathcal{A}$ is a *triangulated* A_∞ category.

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Step 3. Get a "honest" triangulated category by setting $\mathrm{D}^b(\mathcal{A}) = H^0(\mathsf{Tw}\,\mathcal{A})$, that is, objects of $\mathrm{D}^b(\mathcal{A})$ are the same as twisted complexes, but

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- Often, it is better to work with the idempotent completion ("adding direct summands"), denoted $\mathrm{D}^\pi(\mathcal{A})$. In the 1-categorical setting: given a linear category $\mathcal{C}$, a morphism $p : X \to X$ is called idempotent if $p^2 = p$. Let Y be the image of p, and we formally add maps $u : X \to Y$, $v : Y \to X$ such that $u \circ v = \mathrm{id}_Y$, $v \circ u = p$. In this new category, this decomposes $X = Y \oplus \mathrm{Ker}\, p$.

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By $\mathrm{D}^b \operatorname{Fuk} M$ (resp. $\mathrm{D}^\pi \operatorname{Fuk} M$), we mean the above construction applied to $\operatorname{Fuk} M$. The shift functor [1] acts by reversing orientation of L and shifting the grading of L by π.

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Conjecture (Kontsevich)

Let M and $\hat{M}$ be two "mirror" CY n-folds. Then there is an equivalence of triangulated categories

$$\mathrm{D}^\pi\,\mathsf{Fuk}\,M \cong \mathrm{D}^b(\mathrm{coh}\,\hat{M}).$$

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Upshot: we get an immersed Fukaya category $\mathfrak{Fuk}\, M$ and immersed derived Fukaya category $\mathrm{D}^b\, \mathfrak{Fuk}\, M$.

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Example: Suppose $(L_1, E_1, b_1), (L_2, E_2, b_2)$ are embedded Lagrangians intersecting tranversely. Then $L = L_1 \cup L_2$ is an immersed Lagrangian brane with transverse self-intersection with two branches. Then we can take $b_c = b_1 \oplus b_2$, together with b_p for each $p \in L_1 \cap L_2$ which encode how L_1 and L_2 relate to L in $\mathrm{D}^b \mathfrak{Fuk}\, M$.

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Similar to the embedded case, $\mathrm{D}^\pi \mathfrak{Fuk}\, M$ will denote the idempotent completion of the immersed derived Fukaya category.

Open (?) problem: Generalize the construction of $\mathfrak{Fuk}\, M$ to more general immersed Lagrangians.

Stability conditions on triangulated categories

Moduli of vector bundles on Riemann surfaces

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Theorem (Donaldson-Uhlenbeck-Yau)

A vector bundle is stable if and only if it admits a Hermitian-Einstein connection.

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Let $E \to X$ be a holomorphic vector bundle on a Riemann surface.
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Every vector bundle E admits a distinguished filtration

$$0 = E_0 \subset \cdots \subset E_n = E$$

whose quotients are semistable with decreasing slope. This is called a Harder-Narasimhan filtration.

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Since this is a complex number, it has a phase ϕ, and $\phi(E) = \frac{1}{\pi}\,\mathrm{arcot}(-\mu(E))$. This also extends to $\mathrm{coh}\,X$.

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Note also that degree and rank are additive, so Z is actually a group homomorphism

$$Z : K_0(\mathrm{coh}\,X) \to \mathbb{C}$$

such that if $E \neq 0$, then $Z(E) \neq 0$.

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Define $P(\phi)$ to be the full (abelian) subcategory of $\operatorname{coh} X$ whose objects are semistable with phase ϕ. The stable objects of phase ϕ are the simple objects in $P(\phi)$.

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By using the isomorphism $K_0(\operatorname{coh} X) \cong K_0(\mathrm{D}^b(X))$, we can now talk about stable objects in $\mathrm{D}^b(X)$.

Bridgeland stability conditions

Definition

Let $\mathcal{T}$ be a triangulated category, a stability condition on $\mathcal{T}$ is a pair (Z, P) consisting of a morphism $Z : K_0(\mathcal{T}) \to \mathbb{C}$ called the central charge, and for each $\phi \in$ R, a full additive subcategory $P(\phi) \subset \mathcal{T}$ such that

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- if $\phi > \psi$, then $\mathrm{Hom}(P(\phi), P(\psi)) = 0$, and
- For each nonzero object $F \in \mathcal{T}$, there is a finite sequence $\phi_1 > \cdots > \phi_n$ and $A_j \in P(\phi_j)$ such that

$$\begin{array}{ccccccccccccc}
0 = F_0 & \longrightarrow & & F_1 & \longrightarrow & & F_2 \to \cdots \to & F_{n-1} & \longrightarrow & & F_n = F \\
 & {\scriptstyle [1]}\nwarrow & \swarrow & & {\scriptstyle [1]}\nwarrow & \swarrow & & & \nwarrow & \swarrow & \\
 & & A_1 & & & A_2 & & & & A_n &
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Let $\mathrm{Stab}(\mathcal{T})$ be the set of (locally finite) stability conditions on $\mathcal{T}$. This forms a (possibly infinite dimensional) complex manifold. The (semi)stable objects under a given stability condition can be packaged into a "geometric space".

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Often this geometric space is nice (e.g. projective variety or compact complex manifold) of some (likely high) dimension. This gives good tools to study the geometry of higher dimensional spaces by relating them to the study of the derived categories of the lower dimensional spaces. Also can be used to give information about the autoequivalences of $\mathcal{T}$.

Joyce's Conjectures

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Conjecture (Joyce-Thomas-Yau)

Let M be a CY n-fold, either compact or with suitable conditions in the noncompact setting, and let $\mathrm{D}^b \operatorname{Fuk} M$ be the embedded derived Fukaya category. Then there is a Bridgeland stability condition on $\mathrm{D}^b \operatorname{Fuk} M$ such that

(a) *The morphism Z is the composition*

$$K_0(\mathrm{D}^b \operatorname{Fuk} M) \xrightarrow{(L,E,b)\mapsto[L]} H_n(M,\mathbb{Z}) \xrightarrow{[L]\mapsto\int_L \Omega} \mathbb{C}$$

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(c) *Enlarge $\mathrm{D}^b \operatorname{Fuk} M$ to include singular and immersed Lagrangians. Then for every $\phi \in \mathrm{R}$, every isomorphism class of objects in $P(\phi)$ has a unique representative (L,E,b) with L special Lagrangian of phase ϕ.*

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Part (c) above is likely false as stated, simply because it may require more singular Lagrangians then the theory can handle. An alternative is the following:

Conjecture

Enlarge $\mathrm{D}^b \operatorname{Fuk} M$ *so that it contains some immersed or singular Lagrangians. Then for any* $\epsilon > 0$ *and* $\phi \in \mathsf{R}$*, every isomorphism class of objects in* $P(\phi)$ *has a representative* (L, E, b) *with* $\theta_L : L \to (\pi\phi - \epsilon, \pi\phi + \epsilon)$

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Some remarks:

- An example of the kind of enlargement that may work is $\mathrm{D}^b\,\mathfrak{Fuk}\,M$. Note that this enlargement is necessary for uniqueness, as it is possible that the representative could be immersed. In addition, one proposed way of proving the above conjecture is Lagrangian mean curvature flow, in which some badly singular Lagrangians can arise.

Joyce's Conjectures

- By using Kontsevich's HMS, one can hope to construct stability conditions on $\mathrm{D}^\pi \operatorname{Fuk} M$ by instead looking at stability conditions on $\mathrm{D}^b(\operatorname{coh} \hat{M})$. The latter category is significantly better understood, and explicit stability conditions have been constructed when M is a CY n-fold where $n = 1, 2, 3$ ($n = 3$ case depends on a conjecture).

  The HMS equivalence has been proven for elliptic curves and smooth projective hypersurfaces $X \subset \mathbb{P}^n$ of degree $n + 1$.

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Again, take an enlargement of $\mathrm{D}^b\,\mathfrak{Fuk}\,M$ and suppose $\alpha \in \mathrm{R}$ is such that $Z(L)$ does not have phase α for all classes (L, E, b). Since $Z(L)$ is computed from the (co)homology classes $[\Omega]$ and $[L]$, there are at most countably many phases in the image of Z. So we have many choices for α.

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Define $\mathcal{A}_\alpha$ to eb the full subcategory of objects such that the phase function $\theta_L : L \to (\pi\alpha, \pi(\alpha+1))$, and $\overline{\mathcal{A}}_\alpha$ to be the isomorphism-closure of $\mathcal{A}_\alpha$.

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We say that a nonzero object (L, E, b) in $\mathcal{A}_\alpha$ or $\overline{\mathcal{A}}_\alpha$ is stable (semistable) if there is no distinguished triangle

$$L_1 \to L \to L_2 \to L_1[1]$$

with L_1, L_2 nonzero with $\phi(L_1) \geq \phi(L_2)$ (resp. $>$).v

Conjecture (Joyce-Thomas)

$\overline{\mathcal{A}}_\alpha$ is the heart of a bounded t-structure (so both $\mathcal{A}_\alpha$ and $\overline{\mathcal{A}}_\alpha$ are abelian) and under the conjectured Bridgeland stability condition, $P(\beta)$, $\beta \in (\alpha, \alpha + 1)$ consists of semistable objects in $\overline{\mathcal{A}}_\alpha$ with global phase $\pi\beta$.

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Corollary

$\mathrm{D}^b \operatorname{Fuk} M$ is idempotent complete, that is, $\mathrm{D}^\pi \operatorname{Fuk} M \cong \mathrm{D}^b \operatorname{Fuk} M$.

Joyce's program

Let (M, g) be Riemannian, and $i : N \to M$ a (embedded or immersed) submanifold of lower dimension. The mean curvature flow of N is the study of smooth 1-parameter families $i_t : N \to M$, $t \in [0, T)$, satisfying

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Now lets sketch Joyce's proposed program to be the main conjecture.

Joyce's program

Define Z as previous and $P(\phi)$ to either be the full subcategory of objects isomorphic to a special (possibly singular) Lagrangian of phase ϕ, or the full subcategory of objects which are isomorphic ot a Lagrangian with phase bounded around ϕ. Let $F = (L, E)$ be a nonsingular immersed brane. We need to construct a diagram

$$\begin{tikzcd}[column sep=small] 0 \arrow[r, equal] & F_0 \arrow[rr] & & F_1 \arrow[rr] \arrow[dl] & & F_2 \arrow[r] & \cdots \arrow[r] & F_{n-1} \arrow[rr] & & F_n \arrow[r, equal] \arrow[dl] & F \\ & & A_1 \arrow[ul, "{[1]}"] & & A_2 \arrow[ul, "{[1]}"] \arrow[ur, leftarrow] & & & & A_n \arrow[ul] & & \end{tikzcd}$$

where A_j are special Lagrangians of decreasing phase or Lagrangians with decreasing bounded phase.

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Expected to be, in dimension 3, roughly on par with the Poincaré conjecture in terms of difficulty. In addition, this strategum *requires* working with *at least* immersed Lagrangians, as embedded Lagrangians can flow into immersed one in finite time.

Some other conjectures around stability conditions

Theorem (Haiden, Katzarkov, Kontsevich)

Let S be a marked surface of finite type and $M(S)$ the space of marked flat structures on S. Then there is a natural map $M(S) \to \mathrm{Stab}(\mathrm{Fuk}\, S)$ which is bianalytic onto its image.

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Open question: is this map surjective?

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Some other conjectures around stability conditions

It is a surprisingly interesting question to count the number of curves on a CY n-fold X. There are at least three ways:

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All of these are conjecturally equivalent, related to certain properties and features of the stability manifold $\mathrm{Stab}(X) = \mathrm{Stab}(\mathrm{D}^b(\mathrm{coh}\, X))$.

Thank you!