THE DERIVED MCKAY CORRESPONDENCE IN DIMENSION 2

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Abstract. We discuss the proof of the McKay correspondence in dimension two for the affine plane. Along the way, we discuss the various background material involved in the proof, and provide a fairly complete discussion of the derived category of coherent sheaves in the presence of an action by a finite group. An extensive bibliography and dicussion of the literature is also included throughout the paper.

Contents

1. Introduction

The McKay correspondence, named after John McKay, is a bijection between finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ and the extended Dynkin diagrams appearing in the ADE classification. For such a finite group G, the McKay quiver or McKay graph associated to the tautological action of G on $\mathbb{C}^2$ is the quiver whose vertices correspond to the irreducible representations ρ_i of G and which has $a_{ij} \in \mathbb{Z}_{\geq 0}$ edges between the ith and the jth vertex, where the coefficients satisfy:

$$\mathbb{C}^2 \otimes \rho_i \cong \oplus_j a_{ij} \rho_j.$$

A more geometric realization is to form the quotient $\mathbb{C}^2/G$ as a variety (or orbifold). We end up with a surface $\mathbb{C}^2/G$, and since the group acts freely away from the origin, this surface has a single singularity. In honor of Klein who discovered them [29], singularities which are (locally) isomorphic to those obtained by the quotient $\mathbb{C}^2/G$ are called Kleinian singularities. If we then proceed to obtain a nonsingular minimal model, by blowing-up or otherwise, the pre-image of the singular point is a tree of rational curves. This tree has a certain intersection graph, and this graph is precisely a McKay quiver we described above. That they agree and turn out to be the extended Dynkin diagrams in the ADE classification is the miracle here, and this whole circle of ideas is referred to (loosely) as the McKay correspondence.

A different formulation, called the *geometric McKay correspondence*, was formulated by Gonzalez-Sprinberg and Verdier in 1983 [17]. Roughly speaking, the key idea is to notice that on $U = \mathbb{C}^2 \setminus 0$, G acts freely, and so $U \to U/G$ is a principal G-bundle. Using that a resolution of singularities is an isomorphism away from the exceptional set E, we obtain a principal G-bundle on the complement of the of the resolution, $\tilde{U} \setminus E$. One needs to do some work at this stage, but this extends to give an isomorphism of Grothendieck groups, $K(\widetilde{\mathbb{C}^{\not 2}/\mathbb{G}}) \cong K_G(\mathbb{C}^2)$. So here the McKay correspondence becomes an isomorphism between the equivariant K-theory of an Kleinian singularity to the ordinary K-theory of its resolution. In particular, the isomorphism is given explicitly as a so-called K-theoretic integral tranform, given by the composition of morphisms

$$\begin{array}{ccccc} & & K_{G\times \mathbb{1}}(\mathbb{C}^2 \times \widehat{\mathbb{C}^2/G}) & & \\ & \overset{p_1^*}{\nearrow} & & \overset{[-]^G \circ p_{2*}}{\searrow} & \\ K_G(\mathbb{C}^2) & & \overset{\sim}{\dashrightarrow} & & K(\widehat{\mathbb{C}^2/G}). \end{array}$$

Such a composition is called a "Fourier-Mukai" transform, and has revealed itself to a be a functor of fundamental type in many respects.

The story doesn't end here however. The Fourier-Mukai functor was named after Shigeru Mukai, who first recognized the functor's importance when studying derived categories on abelian varieties. So a natural question to ask if whether or not a Fourier-Mukai functor gives an equivalence of derived categories. Indeed we aim to show such an equivalence. This is the so-called *derived McKay correspondence*, first proven by [27], and was later generalized by [3].

Theorem 1.1 (Derived McKay correspondence)**.** *Let G be a finite subgroup of $SL(2, \mathbb{C})$, acting on $\mathbb{C}^2$ via the natural action. Then there is an equivalence of triangulated categories*

$$\mathcal{D}^b(X) \cong \mathcal{D}^b_G(\mathbb{C}^2)$$

between the bounded derived category of coherent sheaves on the crepant resolution X of $\mathbb{C}^2/G$, and the G-equivariant bounded derived category of coherent sheaves on $\mathbb{C}^2$.

This is of course a result which essentially compares the geometry of the quotient to the equivariant geometry of the plane, which provides some justification for the general principle in equivariant geometry. Namely the study of equivariant things on some object, should be the same as the study of normal things on the quotient object. The notion of geometry here seems loose, but it has long been recognized that equipping a space with a sheaf gives rise to genuine geometric structures, in particular incidence structures (a lá algebraic geometry), and hence one can regard the derived category as a genuinely geometric object in this sense. A more insightful person then myself might also say that this correspondence is a statement about the comparison of two resolutions of singularities, one commutative, and the other non-commutative (in the sense of van den Bergh), but we will not elaborate on this at this point in time[1]. Our aim is to prove Theorem 1.1, and introduce the necessary tools as we go.

Let us now give a brief outline of the notes. The first section is split into three parts. Up first is a rough sketch of the classification of the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$, following [8]. These groups were classified by Klein in [29] using a beautiful argument. We however will follow a different route due to Dolgachev [8], which is a purely algebraic argument, and has the benefit of working over (at least in principle) any characteristic. From a larger perspective the two methods do not diverge so much geometrically, but the original has the benefit of elegance. The following subsection provides a short introduction to algebraic groups and actions on varieties, as well as the existence of a quotient by a finite group (in the affine case). The final section forms an introduction to representation theory, which can be skipped until the section on the equivariant derived category, where it becomes essential.

More can be said about the surfaces $\mathbb{C}^2/G$ however, and the following section addresses them in more detail. In 1934, du Val [11] characterized the Kleinian singularities as those which are *crepant*, i.e., resolving them does not affect the canonical class. Such resolutions are always minimal, in the sense that they are the "smallest" nonsingular variety birational to the original. In the same work du Val obtained the description of the minimal resolution as follows. The pre-image of the singular point $0 \in \mathbb{C}^2/G$ under the resolution map is a connected union of projective lines $E_1 \cup \cdots \cup E_r$, each $E_i \cong \mathbb{P}^1$, with self-intersection -2. To each curve one associates a vertex, and two such vertices are connected by an edge if the corresponding curves intersect. The miracle is that these intersection graphs then give the ADE Dynkin diagrams. The modern and convenient approach is to understand these resolutions as suitable fine moduli spaces of objects on the surface, which are in a rough sense tracking the original polyhedra which arose as part of the classification. This allows us inordinate strength when we define the Fourier-Mukai transform later.

While the classical story ends here, we want to go further and see the equivalence of derived categories as in Theorem 1.1. Derived categories were first developed in the thesis of Verdier [45]. While they have long been seen as the correct framework for derived functors and related the homological algebra of algebraic geometry,they have of late been recognized as an invariant of a scheme which (somewhat mysteriously) controls much of the geometry. The next section (and the following) deal exclusively

[1]Maybe sometime in the future?

with the foundations of their theory. This section and the next forms the bulk of the note. I have tried to simultaneously make the discussion elementary and yet comprehensive, and I hope that I have succeeded. The related equivariant derived category is also addressed shortly thereafter, and the final section gives a brief but thorough tour of the theory of Fourier-Mukai transforms before moving on the proof.

2. Algebraic Groups and Geometric Invariant Theory

2.1. Finite Subgroups of $\mathrm{SL}(2,\mathbb{C})$.

Our first goal will be to explain how the following classical theorem comes to be. This is not, strictly speaking, integral to the rest of the note, so the reader may skip this section if they desire. We don't prove the following in complete detail either, so more interested readers should see the references therein.

Theorem 2.1. *Let G be a finite subgroup of $SL(2,\mathbb{C})$. Then up to conjugacy, G is one of the following:*

1. $\mathbb{Z}/n\mathbb{Z}$ *for some* $n \in \mathbb{Z}_{>0}$,
2. *a binary dihedral group of order* $4n$,
3. *the binary tetrahedral group of order 24,*
4. *the binary octahedral group of order 48, or*
5. *the binary icosahedral group of order 120.*

This classification was originally done by Klein, the original argument is briefly as follows. Any finite subgroup of a matrix group is conjugate to some finite subgroup of a unitary group, as any finite subgroup preserves some inner product on the vector space upon which it naturally acts. Passing then to $\mathrm{SU}(2)$, we have at our disposal a beautiful mathematical fact: it is the double cover of $\mathrm{SO}(3)$. This allows us to think about finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ as possible double covers of finite subgroups of the isometry group of the sphere. Such finite subgroups come in only a few classes, namely the symmetry groups of regular polyhedra, and two infinite families given by the cyclic groups of any order and the dihedral groups. Lifting these back to the original group, we obtain the desired classification. Note that by the dual identification of regular polyhedra, we only need to consider the binary groups corresponding to the tetrahedron, octahedron, and icosahedron.

While the above is certainly elegant and beautiful, we will follow roughly the first half of a more algebraic route, due to Dolgachev [8]. This is more natural for our purposes, and at least a priori, has the benefit of working over a field of any characteristic. We know that $\mathrm{GL}(2,\mathbb{C})$ acts naturally on $\mathbb{P}^1$, i.e., $[t_0 : t_1] \mapsto g[t_0 : t_1] = [at_0 + bt_1 : ct_0 + dt_1]$, and if we quotient by $g \sim \lambda g$ for $\lambda \in \mathbb{C}^\times$ we have the well-known fact $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2,\mathbb{C})$. However we can factor this into two steps, namely passing to $\mathrm{SL}(2,\mathbb{C})$ first, then to the projective general linear group. Indeed sending $g \to g/\det(g)$ is a map $\mathrm{GL}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$.

Since we are in dimension two, $\det(g) = \det(-g)$, and so the map sending an element of $\mathrm{SL}(2,\mathbb{C})$ to the automorphism of $\mathbb{P}^1$ it determines as $\pm I_2$ as its kernel. This discussion has proven the proposition below.

Proposition 2.1. *The sequence*

$$1 \to \mathbb{Z}/2\mathbb{Z} \to SL(2,\mathbb{C}) \to Aut(\mathbb{P}^1) \to 1$$

(with maps described in the proof) is exact.

So now given a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, its image is a finite automorphism group of the projective line. Conversely, if we have a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, its order is either even or odd. If the group is even, then it must contain an element of order 2 by Cauchy's theorem, but the only element of order 2 in $\mathrm{SL}(2,\mathbb{C})$ is $-I_2$. So this subgroup arises as the preimage of a finite subgroup of $\mathrm{Aut}(\mathbb{P}^1)$. On the other hand, if the order of the subgroup is odd, then it cannot contain $-I_2$, and so this subgroup is isomorphic to an odd order subgroup of $\mathrm{Aut}(\mathbb{P}^1)$. Thus we have mostly reduced the problem to classifying subgroups of the automorphism group of $\mathbb{P}^1$.

Let $\overline{G}$ be a finite subgroup of $\mathrm{Aut}(\mathbb{P}^1)$. An element of $\mathrm{GL}(2,\mathbb{C})$ has either two one-dimensional eigenspaces or one two-dimensional eigenspace. Indeed the only elements g of finite order which have a two-dimensional eigenspace are those which are scalar multiples of the identity, and hence give the identity automorphism on $\mathbb{P}^1$. On the other hand, the automorphism that g determines will have two fixed points in $\mathbb{P}^1$ corresponding to the two one-dimensional subspaces. Keeping this in mind, set

$$\Sigma = \{(x, g) \in \mathbb{P}^1 \times \overline{G} - \{1\} \mid gx = x\},$$

and $p : \Sigma \to \mathbb{P}^1$ be the projection. Now decompose $p(\Sigma) = O_1 \cup \cdots O_k$, into disjoint orbits. Since all stabilizer subgroups of points in the same orbit are conjugate, $e_i = |\mathrm{Stab}_{\overline{G}}(x)|$ is the same for all $x \in O_i$, so by orbit-stabilizer, $|O_i| = N/e_i$, where $N = |\overline{G}|$.

Counting the cardinality of Σ:

$$|\Sigma| = 2(N-1) = \sum_{x \in p(\Sigma)} (|\mathrm{Stab}_{\overline{G}}(x)| - 1) = \sum_{i=1}^{k} \sum_{x \in O_i} (|\mathrm{Stab}_{\overline{G}}(x)| - 1)$$
$$= \sum_{i=1}^{k} |O_i|(|\mathrm{Stab}_{\overline{G}}(x)| - 1) = \sum_{i=1}^{k} \frac{N(e_i - 1)}{e_i}.$$

Rearranging gives the equation

$$\sum_{i=1}^{k} \frac{1}{e_i} = k - 2 + \frac{2}{N},$$

and since we know that $e_i \geq 2$, the left hand side is less then or equal to $k/2$, and so

$$k - 2 + 2/N \leq k/2 \implies k/2 - 2 + 2/N \leq 0.$$

Since the group is not trivial, $2/n \leq 1$, and so we see that $k = 2$ or 3.

If $k = 2$, then again since $e_i \geq 2$ the only solution is $e_1 = e_2 = N$. Since $|O_i| = N/e_i$, all of $\overline{G}$ fixes two points in $\mathbb{P}^1$. Call these points z_1 and z_2, and by a projective automorphism we may assume they are 0 and ∞. But then in affine coordinates, any such automorphism in $\overline{G}$ must be of the form $z \to az$, for $a^n = 1$ for some n. Hence we arrive at a cyclic group of some order.

If $k = 3$, then things are much more complicated. The equation

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} = 1 + \frac{2}{N}$$

poses the following Diophantine problem: $yzw + xzw + xyw + xyz(w+2) = 0$. The only solutions (in the form (e_1, e_2, e_3, N)) are given below, along with the subgroups (of $\mathrm{SL}(2,\mathbb{C})$) corresponding to them. We could continue at this point, but the analysis is tedious and not too different from the case of $k = 2$ above. The interested reader can see [8] for the full details, and others are welcome to take on faith the result.

It remains however to determine the related structure of the finite subgroups G of $\mathrm{SL}(2,\mathbb{C})$. However referring to our discussion before the analysis, we see that if $\overline{G}$ is not a cyclic group of odd order, then $|G| = 2|\overline{G}|$, and so G is either a cyclic group, binary dihedral group of order $4n$, binary tetrahedral group of order 24, binary octahedral group of order 48, or the binary icosahedron group of order 120. If G is a cyclic group of odd order n, then it can be lifted to an isomorphic copy or to the direct product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

Group	$(e_1, e_2, e_3; N)$	Generators in $\mathrm{SL}(2,\mathbb{C})$
Cyclic $\mathbb{Z}/n\mathbb{Z}$	-	$\begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}$
Binary Dihedral $\mathrm{BD}_{4(n-2)}$	$(2, 2, n, 2n)$	$\begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
Binary Tetrahedral	$(2, 3, 3, 12)$	$\begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{1-i}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$
Binary Octahedral	$(2, 3, 4, 24)$	$\begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{1-i}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$
Binary Icosahedral	$(2, 3, 5, 60)$	$\begin{pmatrix} \zeta_{10} & 0 \\ 0 & \zeta_{10}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5 - \zeta_5^4 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & -\zeta_5 + \zeta_5^4 \end{pmatrix}$

2.2. **Algebraic Groups.** The McKay correspondence as we will present it lies firmly inside the realm of algebraic geometry. As such we want to now tackle the questions of what is a group object in algebraic geometry? what is an action of such an object? and how do we take a quotient of a variety with such an action? The main result of this section is a discussion and proof of the existence of a quotient of an affine variety by a finite (algebraic) group. We follow this with a brief discussion of geometric invariant theory and give the explicit equations which describe the quotient surfaces $\mathbb{C}^2/G$

Definition 2.1. *A group scheme G over k is a group object in the category of schemes over k. That is, we have a "multiplication" $\mu : G \times G \to G$, a morphism for the "inverse" $\iota : G \to G$, and a section $e : \operatorname{Spec} k \to G$ giving the identity element, all satisfying the following diagrams:*

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{id \times \mu} & G \times G \\ \downarrow{\scriptstyle \mu \times id} & & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow{\mu} & G \end{array} \qquad \begin{array}{ccccc} \operatorname{Spec} k \times G & \xrightarrow{e \times id} & G \times G & \xleftarrow{id \times e} & G \times \operatorname{Spec} k \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \cong} & \\ & & G & & \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{\iota \times id} & G \times G & \xleftarrow{id \times \iota} & G \\ \downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{e} & G & \xleftarrow{e} & \operatorname{Spec} k \end{array}$$

Alternatively one could define them in terms of their functor of points, which can be computationally more convenient, but for us obscures the point. An algebraic group will be a group scheme which is reduced, separated, and of finite type over a field (in particular, $\mathbb{C}$). In other words, an algebraic group is a group scheme which is also a variety.

As examples, any finite group is an algebraic group (as we will see below). So is the group of $n \times n$ invertible matrices (indeed, this is the complement of $\det(A) = 0$ in $\mathbb{A}^{n^2}$, which is a polynomial in the matrix entries). As a corollary to this second example, any closed subgroup is also an algebraic group, for a particularly relevant example, $\mathrm{SL}(n, \mathbb{C})$, as this is the group defined by $\det(A) - 1 = 0$.

The most common first examples one encounters are the additive and multiplicative group schemes, given by $\mathbb{G}_a = \operatorname{Spec} k[t]$ and $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, where the multiplication map is given by the maps

$$k[t] \to k[t] \otimes_k k[t], \qquad t \mapsto 1 \otimes t + t \otimes 1$$

and

$$k[t, t^{-1}] \to k[t, t^{-1}] \otimes k[t, t^{-1}], \qquad t \mapsto t \otimes t$$

respectively. We leave it to the reader to check the remaining conditions on the definition.

Another special example is given by $\mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C})$, which is called an (one-dimensional) algebraic torus. More generally, if $T_n \subset \mathrm{GL}(n, \mathbb{C})$ is the subgroup of diagonal matrices, we can easily see that $T_n \cong (\mathbb{C}^\times)^n$, and this is appropriately considered as a higher dimensional algebraic torus. Varieties with an action (on a dense open subset) by an algebraic torus are called *toric varieties*, and these form a particularly amiable class of objects to work with. There is a large amount of literature on the subject of toric varieties.

There are many examples of algebraic groups, however the most important example for us is the following:

Finite Groups

There are many ways to see that finite groups are algebraic. Indeed since $\mathrm{GL}(n, \mathbb{C})$ is an algebraic group, any Zariski closed subset is also an algebraic group. Since $\mathfrak{S}_n$ can be realized as a group of permutation matrices (a set of closed points in $\mathrm{GL}(n, \mathbb{C})$ - hence an algebraic group), and every finite group is a subgroup of some $\mathfrak{S}_n$, it follows that every finite group is algebraic.

Another construction is a special case of so-called *constant group schemes*. Let G be a finite group, we construct a group scheme

$$\mathbb{G} = \coprod_{g \in G} (\operatorname{Spec} \mathbb{C})_g$$

which is simply the disjoint union of points labeled by group elements. We then give produce the group structure by defining the multiplication to be $\mu : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ sending the two components $((\operatorname{Spec} \mathbb{C})_{g_1}, (\operatorname{Spec} \mathbb{C})_{g_2}) \mapsto (\operatorname{Spec} \mathbb{C})_{g_1 g_2}$. We leave it to the reader to show that this is an algebraic group.

Affine algebraic groups are also intimately connected to objects in algebra known as (commutative) *Hopf algebras*. These algebras, in many ways, are the fundamental objects of study in representation theory. We won't delve into this very intricate and fascinating subject here, but roughly speaking Hopf algebras are algebras that are in a sense "self-dual". In particular if one were to write down what an algebra-object would be in a category, you would have a collection of diagrams indicating addition, multiplication, etc. A coalgebra is an object which satisfies all of the same diagrams, but with the arrows reversed. A Hopf algebra is then something which is both a (unital) algebra and a (counital) coalgebra, with some compatibility between the two structures. In our case, if $G = \operatorname{Spec} H$ is an affine algebraic group over a field k, then H is already a k-algebra, and since the spectrum is a contravariant equivalence, the multiplication map $m : G \times G \to G$ gives rise to an opposite map $H \to H \otimes H$, etc. Thus chasing through all of the conditions, we see that the category of affine algebraic groups over k is equivalent to the category of Hopf algebras over k.

We hope the reader has been enjoying themselves so far, as we now turn away from looking at algebraic groups directly, and now focus on how they interact with other algebraic varieties.

Definition 2.2. *A G-variety X is a variety with the action of an algebraic group G: $\sigma : G \times X \to X$, which is a morphism of varieties, satisfying the following two diagrams:*

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{id \times \sigma} & G \times X \\ \downarrow{\scriptstyle \sigma \times id} & & \downarrow{\scriptstyle \sigma} \\ G \times X & \xrightarrow{\sigma} & X \end{array} \qquad \begin{array}{ccc} \operatorname{Spec} k \times X & \xrightarrow{e \times id} & G \times X \\ & \searrow^{\cong} & \downarrow{\scriptstyle \sigma} \\ & & X \end{array}$$

Our goal here is of course to see the existence of a quotient by a finite group (with no conditions on the action!). First, note that a G-variety comes with an obvious action on its sheaf of functions, in particular if f is a regular function then we can define $gf(x) = f(g^{-1}x)$. This will be important later.

Let us now try to formulate a correct notion of a quotient of a variety by some action of an algebraic group. This is a notoriously tricky question, but the machinery that we will use mostly appears in [36] and [7]. In general, quotients in algebraic geometry are difficult to get a good grip on, simply because the Zariski topology does not offer us enough control on the orbits of an algebraic group. As such the subject of Geometric Invariant Theory addresses this in the case of a so-called *reductive* algebraic group; by carefully discarding bad orbits, and defining a quotient in terms of the good ones. The more commonplace solution these days seems to be that one replaces the Zariski topology with a Grothendieck topology (usually the étale topology) and works with quotient stacks instead, which have nice existence and universal properties.

For the reader who wants to learn more, the two sources mentioned above are invaluable. Another introductory treatment can be found in [19], but for us, we can at least motivate what the right definition should be. For starters, we need to know what a quotient of a variety needs to satisfy. The beginning definition is as follows.

Definition 2.3. *Let X be a G-variety with action σ, then a pair $(Y, \pi : X \to Y)$ is said to be a categorical quotient of X by G if the diagram:*

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & Y \end{array}$$

commutes.

Alternatively, we may define a categorical quotient as a pair (Y, π) such that for any G-invariant morphism $f : X \to Z$, there is a unique $g : Y \to Z$ such that $f = g \circ \pi$. This is a perfectly fine definition, but it is a very weak notion of quotient. The fibers $\pi^{-1}(p)$ in general need not be orbits, and even worse the quotient may not have the quotient topology.

We want to define something stronger, typically called the geometric quotient, and the key realization is that we know how to do this as a *set*. We set X/G to be the orbit space, and give it the weakest topology such that the canonical projection $\pi : X \to X/G$ is continuous, i.e., $U \subset X/G$ is open if and only if $\pi^{-1}(U)$ is open. Now we want to give this the structure of a ringed space, so choose some class of functions. If $\phi \in \mathcal{O}_{X/G}(U)$, then the pullback $\pi^*\phi = \phi \circ \pi$ must be a function on $\pi^{-1}(U)$, which is obviously G-invariant. Using this we can make X/G into a ringed space by setting $\mathcal{O}_{X/G}(U) = \mathcal{O}_X(\pi^{-1}(U))^G$. We still want the fibers of π to be orbits however, and still in the realm of sets, this is taken care of by requiring that the map $(\sigma, p_2) = \Psi : G \times X \to X \times X$ having $X \times_{X/G} X$ as its image. Indeed, the elements of $X \times_{X/G} X$ are those pairs in $(x, y) \in X \times X$ each map to the same place under the projection, i.e. $\pi(x) = \pi(y)$. This happens precisely when x and y have the same orbit, so this condition then obviously implies that the fibers of π are the orbits.

In the land of algebraic geometry things are more complicated, as we cannot describe the topology on the fiber product as simply the product topology, yet the same idea still works.

Definition 2.4. *A good geometric quotient of a G-variety X is a pair (Y, π) with $\pi : X \to Y$ a G-invariant morphism satisfying:*

1. *π is surjective,*
2. *for any subset $U \subset Y$, U is open if and only if $\pi^{-1}(U)$ is open in X,*
3. *for any open subset $U \subset Y$, the morphism $\mathcal{O}_Y(U) \to (\pi_*\mathcal{O}_X(U))^G$ is an isomorphism,*
4. *the image of the morphism $\Psi : G \times X \to X \times X$ is $X \times_Y X$.*

Of course, one can check directly that a good geometric quotient is automatically a categorical quotient, so we have not lost anything here. To prove the existence of a good geometric quotient in the case of a finite group, we have one more fact that we wish to recall, this one from commutative algebra.

Proposition 2.2. *Let G be a finite group of automorphisms of a finitely generated integral domain A over $\mathbb{C}$. Then the subalgebra A^G is finitely generated.*

Proof. We first show that A is integral over A^G. Let $b \in A$, and consider the polynomial

$$p_b(t) = \prod_{g \in G} (t - g \cdot b).$$

It is clear that b satisfies this polynomial, that is, $p_b(b) = 0$. Even further, by expanding the finite product of linear factors, this polynomial is monic in t, and it is a general fact that the coefficients of the expanded polynomial are the elementary symmetric functions in the conjugates of b. Almost by definition, the elementary symmetric functions in the conjugates of b lie in the ring of invariants A^G, as applying any element of G simply permutes the terms.

Now since we have shown integrality, let ξ_i be the generators of A as an affine ring over $\mathbb{C}$, and let R be the subring formed by taking the elementary symmetric functions of all conjugates of the ξ_i. By the above, A is integral over R. Thus A is a finitely generated R module, and so we conclude that A^G is a finitely generated A module, and hence an affine ring over $\mathbb{C}$. ■

Finally then we come to the main theorem of this section. The interested reader should seek a more thorough introduction to the subject, as in many places the following theorem is a mere example.

Theorem 2.2. *Let G be a finite group acting on an affine variety X. Then a good geometric quotient X/G exists and is an affine variety.*

Proof. Let $X = \operatorname{Spec} A$, and G be a finite group regarded as an algebraic group. Then we know from above that the ring of invariants A^G is finitely generated over $\mathbb{C}$, so let $Y = \operatorname{Spec} A^G$. We claim that $Y = X/G$. Indeed let $\pi : X \to Y$ be the morphism corresponding to the inclusion $A^G \to A$. We must show that this gives a good geometric quotient. Since we know that A is integral over A^G, the morphism π is surjective by the going-up lemma. Now let $U \in Y$ be open, we may assume it is of the form $D(f)$ for some $f \in A^G$. Then the pre-image $\pi^{-1}(U)$ consists of prime ideals which do not meet the extended ideal $(f)A$, which is clearly open. Conversely, any such open set must arise in this way.

To check the third condition, note that a ring map $f : A \to B$ is an isomorphism if and only if $A_{f^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ is an isomorphism for all primes p. This allows us to see that $\mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$ is an isomorphism by checking the claim on global sections. However this is nearly a tautology, as both the former and latter are equal to A^G, and the map π was induced by the inclusion map $A^G \to A$. Thus the map is the identity, which proves the third condition.

Finally, since the group acts transitively on the primes in X lying over a prime in Y, we must have that $\Psi = (\sigma, p_2)$ has $X \times_Y X$ as its image. This proves that X/G is a good geometric quotient, and is an affine variety defined by A^G. ■

Now for our purposes, the only things that remains are the rings $\mathbb{C}[x, y]^G$ for the finite subgroups G that we classified above. Now $\mathrm{SL}(2, \mathbb{C})$ acts on $\mathbb{C}^2$ as usual, and this induces an action on the functions[2] $\mathbb{C}[x,y]$ via $g.f(x) = f(g^{-1}x)$. In our specific situation, it turns out that $\dim \mathbb{C}^2/G = \dim \mathbb{C}^2 - \dim G$ (see [36] for a more general criterion), and since we are dealing with a finite group, we see that $\dim \mathbb{C}^2/G = 2$. So now all we need are what the invariant polynomials in each case are.

It so happens that the subalgebra is generated by 3 invariant polynomials. Since the quotient $\mathbb{C}^2/G$ should have dimension 2, by Krull's hauptidealsatz the ideal of relations between the 3 generators is principal, and so

$$\mathbb{C}[x,y]^G \cong \mathbb{C}[u,v,w]/(f(u,v,w)).$$

Computing precisely these invariant polynomials varies in difficulty, and to avoid straying to far from the goal we have taken the liberty to include what form f takes explicitly in each case below. As an easy example, the ring of invariants for cyclic groups are simple to compute. Indeed the group $\mathbb{Z}/n\mathbb{Z}$ acts by $x \mapsto \zeta_n x$ and $y \mapsto \zeta_n^{-1} y$. The three invariant monomials can be seen directly to be exactly x^n, y^n, xy, and so we see that

$$\mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{C}[u,v,w]/(uv - w^n).$$

Group	$f(u,v,w)$
Cyclic $\mathbb{Z}/n\mathbb{Z}$	$uv - w^n$
Binary Dihedral $\mathrm{BD}_{4(n-2)}$	$w^2 + u(v^2 + u^n)$
Binary Tetrahedral	$w^2 + u^4 + v^3$
Binary Octahedral	$w^2 + u(v^3 + u^2)$
Binary Icosahedral	$u^2 + v^3 + w^5$

FIGURE 1

2.3. **Representation Theory of Finite Groups.** This section is intended to be a convenient resource for the reader who may have forgotten (or doesn't know) much representation theory. Everything in this section can be skipped for now, but will be used when we discuss the equivariant derived category and briefly when we discuss the G-Hilbert scheme.

Here one should assume that the ground field is $\mathbb{C}$. Let G be a finite group throughout. All of this section was taken from various personal notes of my own, as well as [15] and [32] since many results were borrowed from various places, and the original sources overlap, we have decided to omit citations here, instead directing the reader to the two books mentioned above. We have the following definition, which is the start of the entire subject.

Definition 2.5. *A representation of G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$, where V is some vector space (over $\mathbb{C}$). A finite dimensional representation is a representation where $\dim_{\mathbb{C}} V < \infty$.*

[2]In actuality, we should specify that the group acts on the linear homogeneous polynomials, and then higher polynomials are products of these. That way, we get the needed requirement that $g.(fh) = (g.f)(g.h)$

Strictly speaking, a representation consists of three pieces of data,a group G, a vector space V, and a group homomorphism $\rho : G \to \mathrm{GL}(V)$. However we will simply refer to a representation as the vector space upon which the group acts, and refer to the morphism explicitly when needed. This trades clarity for convenience; we hope this doesn't lead to excessive confusion. Now that we have defined the objects of study, we should give the morphisms to see how they relate to one another.

Definition 2.6. *Let $\rho : G \to \mathrm{GL}(V)$ and $\pi : G \to \mathrm{GL}(W)$ be representations. A morphism $f : V \to W$ is a G-equivariant map of vector spaces (in older terminology, an intertwining map), i.e., a morphism of vector spaces $f : V \to W$ such that the diagram*

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle \rho(g)} & & \downarrow{\scriptstyle \pi(g)} \\ V & \xrightarrow{f} & W \end{array}$$

commutes for all $g \in G$. In terms of elements, this means that $\pi(g)f(v) = f(\rho(g)v)$ for all $g \in G$, $v \in V$. An isomorphism of representations is an isomorphism of vector spaces whose inverse is also G-equivariant.

Here is the first example of such a representation. Let $G \to \mathrm{GL}(V)$ be the morphism sending all of G to the identity endomorphism. This representation is called the n-*trivial* representation, where $n = \dim_{\mathbb{C}} V$. If $n = 1$, then this representation is also called the *trivial* representation. More generally, let us consider a one-dimensional representation of our finite group G (this means that we have a group homomorphism $\rho : G \to \mathbb{C}^{\times}$). Since G is finite, every element has finite order, and so $\rho(g)^n = 1$ for some n depending on g. This immediately shows that all such representations land among roots of unity.

Another important representation is that of the *regular representation*. To define what this representation is, we first need to define the vector space upon which it acts.

Definition 2.7. *Let G be a finite group. The group algebra of G, $\mathbb{C}[G]$, is the $\mathbb{C}$-vector space with basis elements $g \in G$, and $\mathbb{C}$-bilinear multiplication given by*

$$(ag_1 + bg_2)(cg_3 + dg_4) = acg_1g_3 + adg_1g_4 + bcg_2g_3 + bdg_2g_4,$$

where the product g_ig_j is computed in G.

Since G is finite, the group algebra is a finite dimensional vector space of dimension $|G|$. Moreover, any $g \in G$ acts by automorphisms on the left, as given any $x \in \mathbb{C}[G]$, it can be written as

$$x = \sum_{g \in G} a_g g,$$

where g are regarded as formal basis elements. Then

$$g' \cdot x = \sum_{g \in G} a_g g' \cdot g = \sum_{g \in G} a_g (g'g).$$

By definition then, we define the regular representation of G to be the representation (up to isomorphism) corresponding to the left multiplicative action on the group algebra $\mathbb{C}[G]$.

Note that by definition, the translates of 1 in $\mathbb{C}[G]$ are a basis, that is, $\{g \cdot 1 \mid g \in G\}$ are a basis for the group algebra. More generally if $v \in V$ is a fixed vector such that $\{\rho_g(v)\}$ form a basis for V, then $V \cong \mathbb{C}[G]$. An isomorphism $f : \mathbb{C}[G] \to V$ can be given by simply setting $f(g) = \rho_g(v)$ and extending linearly. More generally, if G acts on a set A, denote by V_A the vector space spanned by elements of A. Define a representation $\rho : G \to \mathrm{GL}(V_A)$ by $\rho_g(e_a) = e_{ga}$, where $a \in A$ and ga is the assumed action of G on A. This is called the permutation action associated to G and A. If $A = G$ and G acts by left multiplication, then by definition $V_A \cong \mathbb{C}[G]$.

We would like to relate, in some sense, all representations of G to the regular representation. We can do this in many ways, the perhaps most classical way is using characters, but we are more interested in the related theory of representations of algebras. Similar to that of groups, a representation of an algebra A is an algebra homomorphism $A \to \operatorname{End} V$, where V is some vector space. A morphism of representations is again a morphism of vector spaces commuting with the action. Indeed this gives V the structure of an A module. When $A = \mathbb{C}[G]$, we see that the image of and $g \in G$ is automatically invertible, and so one would suspect that these two notions are equivalent. The following proposition essentially confirms this.

Proposition 2.3. *Let* $\mathrm{Rep}(G)$ *be the category of finite dimensional representations of* G *(where we may drop the assumption that* G *is finite for now), and* $\mathrm{Rep}^+(G)$ *be the category of representations of* G *of arbitrary dimension. Let* $\mathbb{C}[G]$*-Mod be the category of left* $\mathbb{C}[G]$ *modules. Then we have an equivalence of categories* $\mathrm{Rep}^+(G) \cong \mathbb{C}[G]$*-Mod.*

Proof. This is near tautological, but we will sketch it. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then V is a $\mathbb{C}$-vector space with an action of G, so define a $\mathbb{C}[G]$ action by $kg \cdot v = k\rho(g)v$. This then obviously gives a $\mathbb{C}[G]$ module. Taking a morphism of representations by the identity to a morphism of $\mathbb{C}[G]$ moduels is well defined, as the equivariant condition simply says that the morphism is linear in the G-action.

For the reverse, given a $\mathbb{C}[G]$-module, we in particular have that its a $\mathbb{C}$-vector space (forgetting the G-action), and we can simply define the representation by the G-action afforded by the assumption. Since G acts by automorphisms, we are done. The same process as the other direction applies for the morphisms, and we leave it to the reader to show that the composition in both directions is (naturally isomorphic to) the identity. ■

In particular for a vector space V, there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{C}}}(\mathbb{C}[G], \mathrm{End}_{\mathbb{C}}(V)) \cong \mathrm{Hom}_{\mathrm{Grp}}(G, \mathrm{GL}(V)),$$

as a linear map is determined by what it does on a basis, and so a representation of a group G is just a representation of the algebra $\mathbb{C}[G]$. Hence we can discuss representations of groups purely in terms of algebra-theoretic concepts.

As an aside, we can relate representations to a special case of a more general concept, namely that of an equivariant module. For a ring R and a R-module M, both with a G

action, an equivariant module is a module where the multiplication map $R \otimes M \to M$ is equivariant. On elements, this simply means $g(rm) = g(r)g(m)$. INote the special case of $R = \mathbb{C}$ and the action on $\mathbb{C}$ is trivial, we simply get a representation of G on a $\mathbb{C}$-vector space (equivalently, a $\mathbb{C}[G]$ module). Let us formally denote the definition.

Definition 2.8. *Let G be a group acting on a ring R. Then a G-equivariant R-module M is an R-module that has an action by G, such that the multiplication map $R \otimes M \to M$ is G-equivariant: $g(rm) = g(r)g(m)$ for all $r \in R$, $m \in M$, and $g \in G$. A morphism of such modules is a G-equivariant morphism of R modules.*

The category of such equivariant modules will be denoted $R\text{-Mod}_G$. In the scenario where R is a $\mathbb{C}$ algebra, we can go further, indeed we can construct a new ring $R\#[G]$, called the skew group ring, as follows. Consider the vector space $R \otimes_{\mathbb{C}} \mathbb{C}[G]$, with the following product:

$$(r' \otimes g')(r \otimes g) = r'g'(r) \otimes g'g.$$

This is a noncommutative unital ring, with unity $1 \otimes 1$. Then we have the following claim:

Proposition 2.4. *There is an equivalence $R\text{-Mod}_G \cong R\#[G]\text{-Mod}$.*

Proof. Let M be a G-equivariant R module. Now we give M the structure of a $R\#[G]$ module as follows: $(r \otimes g).m = rg(m)$. We check the properties:

$$(r'\otimes g').((r\otimes g).m) = (r'\otimes g').(rg(m)) = r'g'(rg(m)) = r'g(r)g'g(m) = ((r'\otimes g')(r\otimes g)).m.$$

where we have used that M has the compatibility between G and R actions, and $(1 \otimes 1).m = 1 \cdot 1(m) = m$, by the definition of an action. Now for morphisms, we check that

$$f((r \otimes g).m) = f(rg(m)) = rg(f(m)) = (r \otimes g).f(m)$$

as we have assumed that f is R and G linear.

This gives a functor from $R\text{-Mod}_G$ to $R\#[G]\text{-Mod}$. It is easy to see that this functor is essentially surjective, as any $R\#[G]$-module has a natural R-module structure and $k[G]$-module structure given by restriction, namely the R action by $r.m = (r \otimes 1)m$ and the $k[G]$ action by $g.m = (1 \otimes g)m$. Now we need to check that this is a G-equivariant R-module, and for this, we note that $(r \otimes 1)(1 \otimes g) = r \otimes g$ in $R\#[G]$. Now

$$g(rm) = (1 \otimes g)((r \otimes 1)m) = (g(r) \otimes g)m = g(r)(g.m).$$

So this functor is essentially surjective. To see that it is fully faithful, reverse the argument for the functor above to see that any morphism of $R\#[G]$ modules gives a morphism of G-equivariant R modules. ■

Equivariant modules then should be seen as a generalization of representation theory, in some sense they are "nonlinear" representations of G. We will discuss skew group modules in much more detail when we reach the equivariant derived category of the affine plane.

Now that we have thoroughly discussed the fundamental objects at hand, it is time to start discussing how to work with them. Given a group G, a common question is to ask what all of its representations are. This seems at first like a naive question, as certainly there are many ways upon which a group acts on a vector space, but the incredibly surprising fact is that they can all be decomposed into a few fundamental pieces, called simple representations. The notion of simple has not yet been defined, so let us do that now.

Definition 2.9. *Let $\rho : \mathbb{C}[G] \to \mathrm{End}(V)$ be a representation. Viewing V as a $\mathbb{C}[G]$ module, a subrepresentation is a $\mathbb{C}[G]$ submodule of V. V is said to be a simple representation if it is simple as a $\mathbb{C}[G]$ module.*

Here is one example: Let $\mathbb{C}[G]$ be the regular representation, and consider the subspace spanned by the vector $\sum g$. Then clearly this one-dimensional subspace is invariant under the action of G, and is isomorphic to the trivial representation. More generally, given any representation V, take any collection of vectors, and consider the span of those vector and all translates $\rho_g(v)$. This is clearly a subrepresentation of V. Typically, simple representations do not arise naturally in practice, but in many cases a representation of an algebra or group can be broken down into simples pieces, namely as a direct sum. This is the notion of semisimplicity.

Definition 2.10. *Let $\rho : A \to \mathrm{End}_{\mathbb{C}}(V)$ be a representation of an algebra A. We say that ρ is semisimple if*

$$V \cong \bigoplus_{i \in I} S_i,$$

where $S_i \subset V$ are simple representations of A. An algebra A is called semisimple if all representations $\rho : A \to \mathrm{End}_{\mathbb{C}}(V)$ are semisimple.

It is a remarkable fact that no matter how complicated the action may be, it can always be written (in an essentially unique way) as a direct sum of representations which are "building blocks" for all others. Of course the reader may raise their eyebrows at this, simply because a definition without any examples is worthless (one can define many remarkable properties about the zero ring or empty set for example). But nevertheless the notion of semisimplicity is abundant in math. We will see later that the group algebra of a finite group is semisimple, giving at least one nontrivial example.

An important application of the concept of semisimplicity is a fundamental lemma in representation theory, whose usefulness should not be underestimated due to the simplicity of the proof.

Lemma 2.2.1 (Schur's Lemma)**.** *Let V and W be simple representations of $\mathbb{C}[G]$, and $\phi : V \to W$ is a morphism of representations. Then ϕ is either zero or an isomorphism. In particular if $V = W$, then $\mathrm{Hom}_{\mathbb{C}[G]\text{-}Mod}(V, W) \cong \mathbb{C}$.*

Proof. We leave it to the reader to check that given a morphism of representations, both the kernel and image are subrepresentations. This implies (by simplicity) that ϕ must either be zero or an isomorphism. If $V = W$, then $\varphi - \lambda I$ is a linear G-equivariant map of vector spaces, and since our ground field is $\mathbb{C}$, it must have nontrivial kernel. This implies that φ cannot be an isomorphism, so by the first part, it must be zero. Hence the claim. ∎

As an example of the utility of this result, we can give the dimensions all simple representations of finite abelian groups. Indeed if $\rho : \mathbb{C}[G] \to \operatorname{End} V$ is a simple representation of a finite abelian group G, then each endomorphism (isomorphism) ρ_g is a G-equivariant map $V \to V$, as $\rho_g \circ \rho = \rho \circ \rho_g$ for all g. Since this morphism is not zero, it must be a scalar multiple of the identity, which shows that $\rho_g = \lambda I$ for some $\lambda \in \mathbb{C}^\times$. This argument of course applies to any $g \in G$, and so $\mathbb{C}[G]$ must act by scalars. Since V was simple, this gives that V is one-dimensional[3].

Now let us turn back to our original goal in this section, relating the simple representations of G to the group algebra $\mathbb{C}[G]$. In fact, more is true. In a precise sense all simple representations appear in the group algebra. As such let A be an algebra over $\mathbb{C}$. Then we define the regular representation of A to be the homomorphism $A \to \operatorname{End}_{\mathbb{C}}(A)$, taking an element a to the operator which multiplies by a on the left. Note that if we take $A = \mathbb{C}[G]$, this is the same regular representation we had defined earlier.

Proposition 2.5. *A is semisimple if and only if its regular representation is semisimple.*

Proof. Clearly if A is semisimple, then its regular representation is semisimple. Conversely, let V be a vector space with a representation ρ of A. Given any collection of generators of V, $\{v_i\}_{i \in I}$, define the surjection

$$\beta : A^{\oplus I} \to V$$

by sending $(a_i)_{i \in I} \to \sum \rho(a_i) v_i$. Endowing A with its regular representation, this yields V as the homomorphic image of a direct sum of copies of the regular representation. We claim that the homomorphic image of a semisimple representation is also semisimple.

Indeed, since A with its regular representation is semisimple, we can write

$$A^{\oplus I} \cong \operatorname{Ker} \beta \oplus \left(\bigoplus_{i \in J} S_i \right)$$

for some subset $J \subset I$. Indeed this follows from the simple fact that the kernel is a subrepresentation (that you can prove yourself). Thus

$$V \cong A^{\oplus I} / \operatorname{Ker} \beta \cong \bigoplus S_i,$$

and we leave it to the reader to verify that the quotient of representations is again a representation. This shows that V is semisimple, proving the claim. ■

Proposition 2.6. *Let A be a finite dimensional $\mathbb{C}$ algebra. Then the following are equivalent:*

1. *A is semisimple;*
2. *$\sum_i (\dim S_i)^2 = \dim A$, where the sum is over all simple representations of A; and*
3. *$A \cong \oplus_i M_{n_i}(\mathbb{C})$, where n_i are the dimensions of the simple representations of A, and the sum is over all simple representations.*

[3]We note that Schur's lemma here isn't strictly necessary. There are several introductory proofs.

Proof. Clearly, (3) implies (1), so let us now prove that (1) implies (3). Let S_i be the simple representations of A. We then have an surjection (which the reader can prove) of representations:

$$\oplus_i \rho_i : A \to \oplus_i \operatorname{End} S_i.$$

The kernel of this map consists of the elements which act by zero in all simple representations of A. Since A is semisimple, it must split up into a direct sum of simple representations, and an element that acts by zero on all simples spans a submodule of A, but by semisimplicity this submodule must either be zero or isomorphic to direct sums of simple representations, but then this contradicts the surjectivity of the map above. Hence the kernel is zero, and we see that

$$\oplus_i \rho_i : A \to \oplus_i \operatorname{End} S_i$$

is an isomorphism. Since $\operatorname{End} S_i \cong M_{n_i}(\mathbb{C})$, this proves (3).

The same argument shows that (1) implies (2), so now let us show that (2) implies (1). The surjective map above shows that since A is finite dimensional (which was use implicitly up to this point), $\sum_i (\dim S_i)^2 \leq \dim A$, and the assumption on equality then implies that the kernel is zero. Hence (2) implies (1) (via (3)). ■

Note that the proof actually shows that *every* representation of A appears in the decomposition. We however are interested in the case of the group algebra, and the first nontrivial result in representation theory is the following.

Theorem 2.3 (Maschke's Theorem)**.** *The group algebra* $\mathbb{C}[G]$ *is semisimple if and only if* G *is finite.*

Proof. We only prove the "only if" part. Clearly it suffices to show that for a finite dimensional representation $\rho : \mathbb{C}[G] \to \operatorname{End} V$, and $W \subset V$ any subrepresentation, we can find an orthogonal complement W' such that $V = W \oplus W'$. Since V is finite dimensional, choose any orthogonal complement W' to W in V. Clearly as vector spaces $V = W \oplus W'$, but possibly not as a represenation of $\mathbb{C}[G]$. Let $p : V \to W$ be the projection, so that $p|_{W'} = 0$ and $p|_W = I$.

Now define a new morphism,

$$\bar{p} = \frac{1}{|G|} \sum_{g \in G} \rho_g p \rho_g^{-1}.$$

Now since W was a subrepresentation, $\bar{p}|_W = I$ as before, and if we set $\bar{W} = \operatorname{Ker} \bar{p}$, we encourage the reader to check that $\bar{W}$ is also a subrepresentation. It then follows that $V = W \oplus \bar{W}$ as a representation of $\mathbb{C}[G]$, and so the group algebra is semisimple. ■

A near trivial corollary to this is what our original goal was.

Corollary 2.3.1. *Let* G *be a finite group. Then* $\mathbb{C}[G] \cong \oplus_i \operatorname{End}_{\mathbb{C}}(V_i)$ *where the sum runs over all irreducible representations of* $\mathbb{C}[G]$*.*

Strictly speaking, the latter ring in the above corollary should be a direct product (as opposed to the direct sum). As we have not completely shown that a finite group has finitely many simple representations. This follows more or less trivially however, as in the above we see that $\mathbb{C}[G]$ is a finite dimensional algebra, so then the direct sum (or product) must be also. This immediately gives that G has finitely many simple representations. To get a better grip on the precise number however, we need characters.

Definition 2.11. *If $\rho : \mathbb{C}[G] \to \operatorname{End} V$ is a finite dimensional representation of G, then we define the character of V, $\chi_V : G \to \mathbb{C}$, to be the function $\chi_V(g) = \operatorname{Tr}(\rho(g))$.*

It is a classical fact (which can be found in any elementary textbook on the subject) that the characters of simple representations of a group G form a basis in the vector space of $\mathbb{C}$-valued *class functions* on G. A class function is a function which is invariant on all conjugacy classes in G. Hence we see that there are as many simple representations as there are conjugacy classes. Take as before the example of a finite abelian group G. Then there are exactly $|G|$ distinct conjugacy classes, and every simple is one dimensional. Hence $\mathbb{C}[G] \cong \oplus_{|G|}\mathbb{C}$.

2.3.1. *Changing the Group.* We also have a handful of functors which allow us to change the group we are interested in, namely restriction and induction. These will be of import later when we discuss equivariant sheaves and their corresponding derived category. If $H < G$ is a subgroup, and $\rho : G \to \mathrm{GL}(V)$ is a representation of G, then there is an obvious restriction functor Res^H_G, which produces the representation of H: $\rho \circ i : H \to \mathrm{GL}(V)$, where $i : H \to G$ is the inclusion. Clearly this functor is exact, and satisfies an obvious transitivity relation. Further, $\operatorname{Res}^{\mathbb{1}}_G$ is the forgetful functor, and Res^G_G is the identity.

There is an opposite process to this, namely how to take a representation of H and produce a representation of G. This is called the induction functor Ind^G_H. To define this, let $\rho : H \to \mathrm{GL}(W)$ be a representation of H. Then let G/H be the space of left cosets of H, and choose some transversal (i.e. a choice of representatives for each coset) $\{g_\sigma\}_{\sigma \in G/H}$. It follows that for any $g \in G$, we can write $g = g_\sigma h$ for some unique $h \in H$ and unique coset σ. Then we define

$$\operatorname{Ind}^G_H(W) = \bigoplus_{\sigma \in G/H} g_\sigma \otimes W,$$

where $g_\sigma \otimes W = \{g_\sigma \otimes w \,|\, w \in W\}$. To check that this indeed inherits the structure of a representation of G, we need to check a few things. Recall that the coset space G/H is a transitive G-set, where G acts by permuting the cosets by left multiplication. Then given any $w \in \operatorname{Ind}^G_H$, we can decompose this into $w = \sum g_\sigma \otimes w$, and define $gw = \sum g(g_\sigma \otimes w)$ linearly. It remains to define the product $g(g_\sigma \otimes w)$. Since $gg_\sigma \in G$, we can write $gg_\sigma = g_\tau h$ for a unique coset τ and element $h \in H$, so define $g(g_\sigma \otimes w) = g_\tau \otimes hw$, and since H already acts on W, this product is defined. In essence, we are essentially specifying that g acts by permutations on the cosets. Checking that $g'(gw) = (g'g)w$ is left to the reader.

We have one lingering fact to check, namely that the definition of the action gw_σ does not depend on the choice of representatives for the cocycles. Let $\{a_\sigma\}$ and $\{b_\sigma\}$ be two such choices. Then given some $g \in G$, we can write $g = a_\sigma h_a = b_\sigma h_b$, and in particular, we see that $a_\sigma b_\sigma^{-1} \in H$. Since W is a representation of H, this product acts by an automorphism on W, and hence gives a G-equivariant map between the two induced representations.

There is another construction of the induced representation from the perspective of the group algebra which is often useful. A representation of H on W is the structure of a $\mathbb{C}[H]$ module on W. Further, the group algebra $\mathbb{C}[G]$ admits the structure of a $(\mathbb{C}[G], \mathbb{C}[H])$ bimodule, so we can take the tensor product $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$. We then define this to be $\mathrm{Ind}_H^G(W)$. Given any element in this tensor product, we can write it as a sum of pure tensors, so assume our element is pure. Then it can be written as $kg \otimes w$ for some $k \in \mathbb{C}$, $g \in G$ and $w \in W$. Since we know that once we fix a choice of representatives for the cosets, we can write $g = g_\sigma h$, then we see

$$kg \otimes w = kg_\sigma \otimes hw,$$

which is precisely the same action we have defined above. We leave it to the reader to check that these are isomorphic as representations.

Theorem 2.4 (Frobenius Reciprocity)**.** Ind_H^G *is a left adjoint functor to* Res_G^H. *That is,*

$$\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_H^G(V), W) \cong \mathrm{Hom}_{\mathbb{C}[H]}(V, \mathrm{Res}_G^H(W)).$$

Proof. By definition and Hom-$\otimes$ adjunction, we see that

$$\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_H^G(V), W) \cong \mathrm{Hom}_{\mathbb{C}[H]}(V, \mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], W)).$$

Now since $\mathbb{C}[G]$ has a $(\mathbb{C}[G], \mathbb{C}[H])$ bimodule structure, we can define a left $\mathbb{C}[H]$ action on $\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], W)$ via $af(b) = f(ba)$. Since $af(1) = f(a)$, one can check that the morphism $f \mapsto f(1)$ is an isomorphism between $\mathrm{Res}_G^H(W)$ and $\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], W)$. ∎

We will see this again when we discuss equivariant sheaves.

3. Kleinian Singularities and Resolutions of Surfaces

This section and the next form the bulk of these notes. Here we aim to explain many of the geometric features of the quotient $\mathbb{C}^2/G$, mainly through a resolution of singularities. We first give the reader a brief reminder on blow-ups of varieties and resolutions in general before moving on to explain a construction of the resolution which will be important for us later on, the G-Hilbert scheme of points. Indeed the original idea for Mukai was to use a "universal sheaf" on an abelian variety (called the Poincaré sheaf) to induce an equivalence of derived categories, and here the idea is similar: the key step is to realise the resolution as a "moduli space" of certain G-equivariant modules. Our main goal in this section is to discuss this moduli space in detail.

Let X be a variety. For many reasons, one may wish to find a way of removing the singularities without damaging the geometry of the smooth locus. One reason might be the study the singularities themselves, as one obvious classification of singularities is how hard they are to get rid of. As such a resolution of singularities is a pair $(\tilde{X}, \pi)$, with $\tilde{X}$ a nonsingular variety and π a proper birational map which is an isomorphism on the regular points of X. The set-theoretic pre-image of the singular points of X is called the exceptional locus. This resolution is called minimal if for any other resolution of singularities $Y \to X$, this map factors through a map $Y \to \tilde{X}$. A priori, it is not obvious at all that such a resolution would exist, if so, finding one may be incredibly nontrivial. In the case of surfaces however, which is what we are dealing with, there are several positive results, the most general of which was proven by Lipman.

Theorem 3.1 ([31]). *Let X be a two dimensional scheme. Then there is a resolution of singularities $\pi : \tilde{X} \to X$, which can be obtained by a finite sequence of normalizations and blow-ups of closed points. A resolution is minimal if and only if the exceptional locus does not contain a smooth rational curve with self-intersection -1. A minimal resolution is unique up to isomorphism.*

This gives us some hope. Lipman's result not only guarantees the existence of a minimal resolution, it gives a method to find it, and a method to test if any resolution is minimal. Hardly something to sneeze at.

Recall that if $X = \operatorname{Spec} A$ is an affine scheme, a closed subscheme Z defines an ideal $I \subset A$. We can then construct the *blow-up along Z* to be the scheme defined by

$$B_Z(X) = \operatorname{Proj}\left(\bigoplus_{n\geq 0} I^n\right)$$

where $I^0 = A$, and the algebra multiplication is induced by the multiplication in A. Note that the blow-up has a natural morphism $\pi : B_Z(X) \to X$, induced by the inclusion $A \to A \oplus I \oplus I^2 \oplus \cdots$ and so $B_Z(X)$ is naturally a X-scheme. This morphism can be shown to be proper and birational away from Z [20], and so the blow-up (assuming that $B_Z(X)$ is nonsingular) gives a resolution of singularities. If the blow-up is not smooth, we may need to repeat the process. The scheme-theoretic preimage of Z is defined to be the closed subscheme defined by the pullback of the ideal sheaf, $\pi^*(\mathcal{I})$, and is denoted E_Z. It is a fact that the blow up comes with a sheaf like $\mathcal{O}(1)$,and the invertible sheaf on $B_Z(X)$ defining E_Z is $\mathcal{O}(-1)$. The reduced cartier divisor $(E_Z)_{red}$ is by the definition the exceptional divisor of the blow-up.

Computing the blow-up can be hard, but if we stick to blowing up at a smooth subscheme, we get a good characterization of the Ree's algebra. It turns out that if $I = (g_1, ..., g_n)$ is generated by a regular sequence, then $\bigoplus_{n\geq 0} I^n \cong A[a_1, ..., a_n]/\tilde{I}$, where $\tilde{I}$ is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} g_1 & \cdots & g_n \\ a_1 & \cdots & a_n \end{pmatrix}.$$

This is illustrated in the following example.

Example:

For example, consider the blow up of $\mathbb{A}^n = \operatorname{Spec} k[x_1, ..., x_n]$ at the origin, which is defined by the maximal ideal $I = (x_1, ..., x_n)$. If we set $S = \oplus_n I^n$, the blow up is then given by $\operatorname{Proj} S$. To figure out what this ring looks like, set $A = k[x_1, ..., x_n]$, and consider the map of graded rings[a] $A[a_1, ..., a_n] \to S$ sending $a_i \mapsto x_i \in I$. Then we see that the kernel is then generated by the homogeneous relations $x_i a_j - x_j a_i$ for $i, j = 1, ..., n$. The projection map gives the map $\pi : \operatorname{Proj} S \to \mathbb{A}^n$.

Now consider the inverse image of the origin, $\pi^{-1}(0)$. Look at the affine chart where a_i does not vanish, we get relations $x_j = a_j x_i$,and hence the ideal pulls back to (x_i), which is principal.

[a]The reason why we consider this ring is that the blow-up is easily realized as a subset of the product $\mathbb{P}^{n-1} \times \mathbb{A}^n$.

So then the reader may ask, if blowing up at smooth points is easier, how do we tackle the singular cases? This is the notion of embedded resolution. While a given variety X may be singular at some point $x \in X$, it may happen that X lives inside a larger ambient space, where x (regarded as a point in the ambient space) is a smooth point. The idea is then to blow-up this larger space at x, and see what happens to X under this birational map. This is called the strict transform of X, and the following proposition guarantees its existence.

Proposition 3.1 ([20]). *Let $f : Z \to X$ be a closed immersion and $\mathcal{I}$ be a sheaf of ideals on X, and $\mathcal{J} = f^{-1}\mathcal{I}\mathcal{O}_Z$ the inverse image ideal sheaf on Z. Let $\tilde{X}$ be the blow-up of X along $\mathcal{I}$ and $\tilde{Z}$ be the blow-up of Z along $\mathcal{J}$. Then there is a unique map $\tilde{f}$, which is a closed immersion, making the diagram commute:*

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \pi_X} \\ Z & \xrightarrow{f} & X \end{array}$$

This is of particular interest to us, as in our context Z will be our singular surface $\mathbb{C}^2/G$, and $X = \mathbb{A}^3$. In this case we will call $\tilde{Z}$ the strict transform of Z in X. To illustrate this, consider the case of $G = \mathbb{Z}/n\mathbb{Z}$ below.

Example:

Consider the surface $xy - z^n = 0$ in $\mathbb{A}^3$. We know that this has an isolated singularity at the origin and nowhere else. Further, we have already seen that the blow-up of affine 3-space is given by

$$\pi : \tilde{X} = \operatorname{Proj} \frac{\mathbb{C}[x, y, z][a, b, c]}{(xb - ya, xc - za, yc - zb)} \to X,$$

where x, y, z have degree zero and a, b, c have degree 1. Now to find what our surface has become (i.e., the strict transform of $\mathbb{C}^2/G$ in the blow-up), we look at an affine chart and use the defining relation $xy - z^n$.

(1) In the chart $a = 1$, where the blow-up is given by

$$\operatorname{Spec} \mathbb{C}[x, y, z][b, c]/(xb - y, xc - z) = \operatorname{Spec} \mathbb{C}[x, b, c],$$

the relation $xy - z^n$ becomes $x(bx) - (cx)^n = x^2(b - c^n x^{n-2})$. The x^2 defines the inverse image of the origin, and we see that our surface intersects the copy of $\mathbb{P}^2$ with multiplicity 2 (hence why these singularities are sometimes called rational double points), while the strict transform is defined by $b - c^n x^{n-2}$.

(2) In $b = 1$ we get a similar story. The strict transform is given by $a - c^n y^{n-2}$.
(3) Similarly for $c = 1$, we get $ab - z^{n-2}$.

This last chart ($c = 1$) is what interests us in particular, as we can check that this is nonsingular if $n \leq 3$, and otherwise we get a singularity in this chart of the same type, but with a lower exponent. If we then blow-up the strict transform up again, the same logic yields a surface which is either nonsingular or closer to nonsingular, and so we see that after a finite number of iterations we get a resolution of singularities. Further, since the intersection of the strict transform of this surface meets the exceptional divisor with multiplicity two, we see that the exceptional divisor in the strict transform has self-intersection -2. Thus this resolution is *minimal*.

3.1. **Hilbert Schemes.** Now we will pause the discussion of birational geometry to discuss the Hilbert scheme of points. This is a fundamental object in algebraic geometry for many reasons, and in our case forms a very interesting example of a resolution of singularities.

The Hilbert scheme of points of an algebraic variety is itself a scheme with parametrizes collections of points with multiplicities in such a way that it satisfies a very convenient universal property. More generally, for a variety X over $\mathbb{C}$ we can define a functor

$$\mathrm{Hilb}^n_X(B) = \begin{Bmatrix} \text{Closed subschemes } Z \subset X \times B, \text{ flat over } B, \text{ whose fibers} \\ Z_b = \pi_B^{-1}(b) \text{ have Hilbert polynomial } n \end{Bmatrix}.$$

Note that if the base is reduced, then this is equivalent to requiring that the Hilbert polynomial is constant among fibers [20]. Further, the requirement that the Hilbert polynomial takes the constant value n, and the subscheme be flat means that $(\pi_B)_*(\mathcal{O}_{Z_b})$ is locally free of rank n on B. The Hilbert scheme is then the scheme which represents this functor.

Theorem 3.2 (Grothendieck)**.** *The functor* Hilb^n_X *is represented by a scheme of finite type over* k*, denoted* $\mathrm{Hilb}^n X$*, and is called the Hilbert scheme of points on* X*. Moreover if* X *is projective, then so is* $\mathrm{Hilb}^n X$*.*

Recall that the notion of representability is that the S-points of $\mathrm{Hilb}^n X$ are in bijection with the objects of the image of the functor $\mathrm{Hilb}^n_X(S)$. In particular, the geometric points $\mathrm{Hilb}^n X(\mathbb{C}) = \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}, \mathrm{Hilb}^n X)$ is in bijection with subschemes Z in X of dimension 0 and length n. By length, we of course mean the length of zero-dimensional Artinian ring $\mathcal{O}_{X,Z}$. It is a general fact that any such representable functor comes with a universal object, that is, an element of $\mathrm{Hilb}^n_X(\mathrm{Hilb}^n X)$, denoted $\mathcal{Z}_n$, such that any other flat family of subschemes satisfying the same conditions can be expressed as the pullback of this family. That is, if we have a flat family $p : Z \subset X \times B \to B$, then by representability this corresponds to a unique morphism $f : B \to \mathrm{Hilb}^n X$, as in the diagram:

$$\begin{array}{ccc} Z \subset X \times B & \xrightarrow{f^*} & \mathcal{Z}_n \\ \big\downarrow p & & \big\downarrow \pi \\ B & \xrightarrow{f} & \operatorname{Hilb}^n X \end{array}$$

then there is a unique morphism $f^* : Z \to \mathcal{Z}_n$ making the diagram commute, and $Z \cong \mathcal{Z}_n \times_{\operatorname{Hilb}^n X} B$. This universal object is what will play an essential role in the proof of the derived McKay correspondence as we mentioned at the beginning of this section.

If we are on an affine variety, then we can describe the Hilbert scheme of points as a set in a more explicit manner.

Definition 3.1. *Let $X = \operatorname{Spec} A$ be an affine algebraic variety. Then the Hilbert scheme of n points in X is defined by*

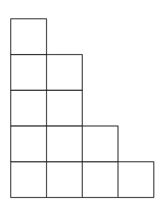
$$\operatorname{Hilb}^n X = \{J \subset A | J \textit{ is an ideal in } A, \dim_{\mathbb{C}} A/J = n\}.$$

Note that we have specified that the dimension of A/J, as a vector space over $\mathbb{C}$, is n. This means that J has maximal height (where height for a nonprime ideal is defined as a the height of a minimal prime containing it), and hence is the defining ideal for some collection of points with multiplicity. The weighted sum of the points with their associated multiplicities should then equal n. This agrees with the definition we gave earlier when we specified that the subscheme have finite *length*. For example, the Hilbert scheme on one point is simply the collection of maximal ideals in A. The Hilbert scheme on two points parametrizes pairs of points, but also single points and a tangent vector (a "velocity"). Indeed one should think about closed points of the Hilbert scheme as simply the set of points of length n, as described above. However non-closed points and even subschemes of the Hilbert scheme of points should be thought of as families of such points, and the Hilbert scheme encodes all possible families in a nice way.

Before we delve into a sketch of the quasi-projectivity of the Hilbert scheme however, we want to introduce a related object, the symmetric product. Let $X = \operatorname{Spec} A$ be an affine variety over $\mathbb{C}$. Then the nth symmetric power of X is defined to be $S^n X = X^n/S_n = \operatorname{Spec}(A^{\otimes n})^{S_n}$. For example if $X = \operatorname{Spec} \mathbb{C}[x]$, then $S^n X = \operatorname{Spec} \mathbb{C}[x_1, ..., x_n]^{S_n}$, and by standard theory of symmetric functions, equal to $\operatorname{Spec} \mathbb{C}[s_1, ..., s_n] = \mathbb{A}^n$. However in general the symmetric power is very singular, and we will see later that this is closely related to the Hilbert scheme of points.

Now we sketch the proof of the quasi-projectivity of $\operatorname{Hilb}^n X$ in the case of $X = \mathbb{C}^2$ following [18], who gives a very different proof then the usual (see for example, [13]).

We first sketch what affine varieties form the open cover. Choose a partition of μ of n, and consider the Ferrers diagram for that partition. For example if $n = 12$, and we have $\mu = (1, 2, 2, 3, 4)$, the Ferrers diagram is



We then endow each of the boxes with x, y-coordinates, with the bottom left square having coordinates $(0,0)$. Based on those coordinates $(i,j) \in \mu$, we then consider the set $B_\mu = \{x^i y^j \,|(i,j) \in \mu\}$. In our example above,

$$B_{(1,2,2,3,4)} = \{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4\}.$$

Now we define the subset of the Hilbert scheme:

$$U_\mu = \{I \in \operatorname{Hilb}^n \mathbb{A}^2 \,|\, B_\mu/I \text{ spans } \mathbb{C}[x,y]/I\}.$$

We get for free that B_μ/I is actually a basis[4], as the $\mathbb{C}$-dimension of $\mathbb{C}[x,y]/I$ is n, and $|B_\mu| = n$. So then for any monomial $x^h y^k$, we can write

$$x^h y^k = \sum_{(i,j)\in\mu} f_{ij}^{hk}(I) x^i y^j \bmod I$$

for some functions f_{ij}^{hk} which depend on I.

Proposition 3.2 (Proposition 2.1, [18])**.** *The sets U_μ are open affine varieties which cover $\operatorname{Hilb}^n \mathbb{C}^2$. The structure sheaf of each of them is generated by the f_{ij}^{hk} for all $(i,j) \in \mu$ and all (h,k).*

The key step in the proof of this result is a basic fact about monomial ideals: for every ideal I in a polynomial ring, there is a basis B mod I, consisting of monomials, such that every divisors of a monomial in B is also in B. Then for any ideal I in $\operatorname{Hilb}^n \mathbb{C}^2$ it is clear that such a basis must be B_μ for some partition μ of n.

Now since $\mathbb{C}[x,y]/I$ has one of the B_μ as a basis, we see that the set of monomials of degree at most n spans this vector space, and hence I determines a map from $\operatorname{Hilb}^n \mathbb{C}^2$ into the Grassmann variety of n-dimensional quotients of this space of monomials. Then a difficult argument of Grothendieck shows that this map is injective and locally closed, hence giving the Hilbert scheme of points in $\mathbb{C}^2$ the structure of a quasi-projective scheme. Note that this scheme is not projective, nor is it even a variety, as it can have nonreduced structure. There is an example, due to Mumford, where an entire connected component of a Hilbert scheme is nonreduced. However, the Hilbert scheme is always connected, which was shown in Hartshorne's thesis.

The relation between the Hilbert scheme of points and the symmetric product that we had hinted at earlier is nicely presented as the following theorem.

Theorem 3.3 (Theorem 11.2, [28] or the discussion in [26])**.** *Let $X = \operatorname{Spec} A$ be an affine variety over $\mathbb{C}$.*

(1) *There is a canonical projective morphism (the Hilbert-Chow morphism) $\pi : \operatorname{Hilb}^n X \to S^n X$, which sends*

$$J \mapsto \sum_{p \in supp(A/J)} \dim_{\mathbb{C}}(A/J)_{\mathfrak{m}_p} \cdot p.$$

We recall the support of an A-module M is the set of primes $\mathfrak{p} \subset A$ where $M_{\mathfrak{p}} \neq 0$.

[4]By B_μ/I, we mean the set of images of the elements of B_μ under the canonical projection $\mathbb{C}[x,y] \to \mathbb{C}[x,y]/I$.

(2) *Let $S^n_0 X$ be the subset of distinct n-tuples of closed points of X^n up to the action of S_n. Denote by $\operatorname{Hilb}^n_0 X$ the pre-image of $S^n_0 X$ by the Hilbert-Chow morphism. Then the restriction $\pi : \operatorname{Hilb}^n_0 X \to S^n_0 X$ is an isomorphism. Further if $\dim X = 2$, then $\operatorname{Hilb}^n_0 X$ is dense in $\operatorname{Hilb}^n X$.*

Proof. (1) The statement of the morphism should be clear. A detailed construction of the morphism and its projectivity is left to the literature (for example, [36]).

(2) It should be clear from the definition that if $Z \in \operatorname{Hilb}^n X$ is any zero-dimensional subscheme which is reduced, then Z is a union of n points. Thus over $S^n_0 X$ the Hilbert-Chow morphisms is the identity. Since $\operatorname{Hilb}^n X$ is connected, and $S^n_0 X$ is open and dense in the symmetric product, this proves the claim.

■

We also have the following important result due to Fogarty [14]:

Theorem 3.4 (Fogarty)**.** *If X is a nonsingular surface, then $\operatorname{Hilb}^n X$ is nonsingular, and the Hilbert-Chow morphism $\pi : \operatorname{Hilb}^n X \to S^n X$ is a resolution of singularities.*

There are a few places one can go for the proof of this result. In the case where X is projective, [41] has a very readable proof. Another more general proof can be found in [37], and of course the original at [14]. A complete and very readable proof in the affine case is done in [8]. Let us examine a particular case in detail, $X = \mathbb{A}^2$, which as we will see is the exact setting that we find ourselves in.

Example:

We first wish to describe the symmetric product $S^2 X$. We can realize the product as $\operatorname{Spec}(\mathbb{C}[\lambda_1, \mu_1, \lambda_2, \mu_2])$, where the action of $S_2 = \mathbb{Z}/2\mathbb{Z}$ is given by $\lambda_1 \leftrightarrow \lambda_2$ and $\mu_1 \leftrightarrow \mu_2$. If we then make the change of coordinates,

$$x_1 = \lambda_1 + \lambda_2, \quad y_1 = \mu_1 + \mu_2, \quad x_2 = \lambda_1 - \lambda_2, \quad y_2 = \mu_1 - \mu_2,$$

then $S^2 X$ becomes isomorphic to $\operatorname{Spec}(\mathbb{C}[x_1, y_1] \otimes_{\mathbb{C}} \mathbb{C}[x_2, y_2]^{\mathbb{Z}/2\mathbb{Z}})$, where $\mathbb{Z}/2\mathbb{Z}$ acts trivially on $\mathbb{C}[x_1, y_1]$ and acts by negation on $\mathbb{C}[x_2, y_2]$. However we already know that the action of $\mathbb{Z}/2\mathbb{Z}$ yields a hypersurface in $\mathbb{A}^3$ by virtue of the ring of invariants, that is, $\mathbb{C}[x_2, y_2]^{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{C}[u, v, w]/(uv - w^2)$, a singular quadric surface. Hence $S^2 X = \mathbb{A}^2 \times Q$, where Q is a quadric cone.

Now given $(t_1, t_2) \in S^2_0 X$, then $\pi^{-1}((t_1, t_2)) = J_{(t_1, t_2)} = \{f \in \mathbb{C}[x, y] \mid f(t_1) = f(t_2) = 0\} = \mathfrak{m}_{t_1}\mathfrak{m}_{t_2}$ is a single point. Indeed since the two points are distinct, their corresponding maximal ideals are comaximal, and so by the Chinese Remainder Theorem $\mathbb{C}[x, y]/J \cong \mathbb{C} \oplus \mathbb{C}$. By definition, the Hilbert-Chow morphism takes J to $\operatorname{supp}(\mathbb{C}[x, y]/J)$, which again is by definition the set of primes for which the localization does not vanish. Since a prime is in the support if and only if it contains the annihilator of the module, we see that the only elements of the support are the prime ideals containing J, which are precisely $\mathfrak{m}_{t_1}$ and $\mathfrak{m}_{t_2}$. This verifies the claim that π was an isomorphism when restricted to $\operatorname{Hilb}^n_0 X$ for this case.

The more interesting case is $t_1 = t_2$, which we may as well assume is given by the point $(0, 0)$. Instead of describing the ideal $\pi^{-1}((0, 0))$, lets instead describe the module $\mathbb{C}[x, y]/J$. We know that this must have dimension 2 as a vector space over

$\mathbb{C}$, be zero dimensional as a ring, and has support at the origin with multiplicity 2. So then as a module over $\mathbb{C}[x,y]$, the variables x, y must act nilpotently, and one can convince themselves that any such module must be of the form

$$\mathbb{C}[x,y]/J = \mathbb{C}[x,y]/(x^2, y^2, xy, ax+by),$$

for some $(a,b) \neq (0,0)$. Hence we have an entire $\mathbb{P}^1$ worth of pre-images. Since we know that this is a resolution of singularities, we can conclude that $\operatorname{Hilb}^2 X = \hat{Q} \times \mathbb{A}^2$, where $\hat{Q}$ is the resolved quadric cone.

3.1.1. *G-Hilbert Schemes.* Our original goal here was to construct a resolution of the singular surface $\mathbb{C}^2/G$, and to do this we need to look at scheme which is closely related to the Hilbert scheme above, the so-called *G-Hilbert scheme*. In particular we will not work in the generality that we sometimes worked with above, and only consider $\operatorname{Hilb}^n \mathbb{C}^2$ from now on. Further we will be dealing with a finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$, and so we always assume that $n = |G|$, whenever such an assumption makes sense.

The action of G on $\mathbb{C}^2$ naturally induces an action on $\operatorname{Hilb}^{|G|} \mathbb{C}^2$ and $S^n\mathbb{C}^2$ by simply acting on each point individually. If we consider the fixed point locus of this action, we have the so-called G-Hilbert scheme[5]: $(\operatorname{Hilb}^{|G|} \mathbb{C}^2)^G = \operatorname{Hilb}^G \mathbb{C}^2$. The corresponding fixed point locus of $S^n\mathbb{C}^2$ is denoted $(S^n\mathbb{C}^2)^G$. Of course, for the G-Hilbert scheme to be a resolution, it must be smooth. However its a standard fact in the setting of a reductive group G acting on a smooth variety, the fixed locus X^G is smooth.

Proposition 3.3. $\operatorname{Hilb}^G \mathbb{C}^2$ *is a nonsingular algebraic variety.*

For a proof, see [8] Lemma 6.2.3 for a particularly readable one.

Note that for any $J \in \operatorname{Hilb}^G \mathbb{C}^2$, the quotient $\mathbb{C}[x,y]/J$ is a representation of G, and hence can be decomposed into simple representations by Maschke's Theorem. This implies that one can to decompose the G-Hilbert scheme into a disjoint union of subschemes according to the decomposition into simple representations [28]. One particular subscheme of interest is the connected component corresponding to the regular representation. Recall that regular representation of a group is by definition the representation of the group G on its group algebra $\mathbb{C}[G]$ given by $\mathbb{C}$-linear multiplication of group elements, i.e., the map $G \to \mathrm{GL}(|G|, \mathbb{C})$ sending G to an appropriate group of permutation matrices. For example if $z \in \mathbb{C}^2 \setminus \{(0,0)\}$, consider the ideal J_z of all functions which vanish on the orbit $O_G(z)$ of z under G. Since for nonzero z the stabilizer in G is trivial, the orbit consists of $|G|$ distinct points, and the action of G simply permutes them. Thus $\mathbb{C}[x,y]/J_z$ is isomorphic to the regular representation of G. We will denote the connected component of $\operatorname{Hilb}^G \mathbb{C}^2$ consisting of ideals which give the regular representation by $\operatorname{Hilb}^G_{reg} \mathbb{C}^2$. It turns out that this is our candidate fo the

[5]In more general situations, the definition of the G-Hilbert scheme that we have taken is not the standard one. The actual definition is as follows: The G-Hilbert scheme of X is the irreducible component $\operatorname{Hilb}^G X$ of $\operatorname{Hilb}^n X$ that contains an orbit of some point $x \in X^{reg}$, where

$$X^{reg} = X \setminus \cup_{1 \neq g \in G} X^g,$$

where X^g is the fixed point locus of the action of $g \in G$. It is the largest open subset upon which G acts freely. It is then a conjecture of Nakamura that this coincides with the definition that we have taken.

resolution of the surface $\mathbb{C}^2/G$. To see this, we first relate the surface to the symmetric power in a surprising way.

Proposition 3.4 ([28]). *There is an isomorphism* $(S^n\mathbb{C}^2)^G \cong \mathbb{C}^2/G$, *where* $n = |G|$.

Proof. Since the stabilizer of every nonzero point in $\mathbb{C}^2$ is trivial, every G-orbit is either zero or $|G|$ distinct points. This gives a map $(S^n\mathbb{C}^2 \setminus \{(0,0)\})^G \to (\mathbb{C}^2 \setminus \{(0,0)\})/G$ which sends a collection of points to the class of their orbit. It is trivial to see that this is an isomorphism. This extends to a bijective morphism $(S^n\mathbb{C}^2)^G \to \mathbb{C}^2/G$ (hence birational), which since $\mathbb{C}^2/G$ is normal, is an isomorphism by Zariski's main theorem. ■

We can use this proposition in the following way. If $\pi : \operatorname{Hilb}^n \mathbb{C}^2 \to S^n\mathbb{C}^2$ is the Hilbert-Chow morphism, restricting it to the G-fixed points gives a map (which we may as well call the G-Hilbert-Chow morphism) $\operatorname{Hilb}^G \mathbb{C}^2 \to (S^{|G|}\mathbb{C}^2)^G \cong \mathbb{C}^2/G$. If we then restrict it further, we get a map $\pi_G : \operatorname{Hilb}^G_{reg} \mathbb{C}^2 \to \mathbb{C}^2/G$. We then have the all important theorem:

Theorem 3.5 (Theorem 12.7, [28] or Theorem 9.3, [26]). *The G-Hilbert-Chow morphism* $\pi_G : \operatorname{Hilb}^G_{reg} \mathbb{C}^2 \to \mathbb{C}^2/G$ *is a crepant resolution of singularities.*

We currently are not in a position to prove this theorem, as we need to still discuss the symplectic structure and the notion of a crepant resolution, but regardless this result implies a very interesting interpretation of this entire correspondence. Recall that G was a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, which was the symmetry group of some higher dimensional regular polyhedron. Now the points of $(S^n\mathbb{C}^2)^G$ which are not zero in any coordinate are simply the orbits of G, i.e. *they are the vertices of such a polyhedron.* Under the isomorphism $(S^n\mathbb{C}^2)^G \cong \mathbb{C}^2/G$, we see that the quotient surface $\mathbb{C}^2/G$ can be though of as a geometric space which parametrizes the possible regular polyhedra for which G is the symmetry group. The singular point should be though of as when the polyhedra have zero diameter, and the fact that there are many ways for the polyhedra to "shrink" to zero reflects the singularity.

The points of $\operatorname{Hilb}^G_{reg} \mathbb{C}^2$ are the collections of subschemes of length $|G|$ of which G acts as in the regular representation. So again we can interpret the individual points as representation the entire polyhedron, and as we approach the singular point in $\mathbb{C}^2/G$, the Hilbert scheme keeps track of the direction, tangents, etc., so as to capture the various behavior of the origin.

This is (at least in my opinion) a remarkably clear way of seeing the geometry of this correspondence. To see more details however, we will need to pause to again discuss birational geometry, in particular the notion of a crepant resolution.

3.2. **Generalized Divisors and Crepant Resolutions.** Let X be an irreducible variety. If X is normal, we can define a Weil divisor to be a formal linear combination of prime divisors, where a prime divisor is just a codimension one irreducible subvariety. The correct notion of equivalence for Weil divisors is that of linear equivalence, i.e., two divisors are equivalent is their difference is a principal divisor – the vanishing locus of a regular function. We have another notion of divisors, namely Cartier divisors which are given in terms of invertible sheaves. For the two notions of divisors to coincide, we need the variety to be factorial, i.e. all local rings should be unique factorization domains. We immediately have a problem though, namely that while factorial rings are normal, the converse is not true. So it is possible to have a normal variety where the notion of Weil and Cartier divisors do not coincide. This is troubling for many reasons, as the calculus of sheaves provides an enormous number of tools which simplifies calculations and allows great flexibility in constructions. Thankfully we are able to combine the two notions of sheaves into what are called *divisorial* sheaves, which are able to capture both notions of divisors simultaneously, and still provide nice functorial properties.

Given a coherent sheaf $\mathcal{F}$ on a quasi-projective variety (actually any scheme works for this definition), we can define the dual sheaf[6] to be $\mathcal{F}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. As a consequence of the definition, there is a natural map from $\mathcal{F}$ to the double dual $\mathcal{F}^{\vee\vee}$, which sends a section to the evaluation morphism which evaluates morphisms on that section. If this map is an isomorphism, we say that the sheaf $\mathcal{F}$ is *reflexive*. Note that a reflexive sheaf is automatically torsion free, and further coherent sheaves (on reasonable schemes) always have a nonempty open subset upon which the restriction is locally free. Thus we can talk about the *rank* of these sheaves, defined as the rank of their restrictions to these open subsets. Further, in the case of torsion free (and a normal variety) we can actually guarantee that the complement of this open set has codimension at least 2. We state for convenience the following result, for the proof one should consult [25].

Proposition 3.5 (Theorem 5.1.11 [25]). *For a torsion free coherent $\mathcal{O}_X$-module $\mathcal{L}$ on a normal variety, the following are equivalent.*

1. *$\mathcal{L}$ is reflexive;*
2. *If a torsion free $\mathcal{O}_X$-module $\mathcal{M}$ satisfies $\mathcal{L} \subset \mathcal{M}$ and $\operatorname{codim}_X(\operatorname{supp} \mathcal{M}/\mathcal{L}) \geq 2$, then $\mathcal{M} = \mathcal{L}$;*
3. *if X_0 is an open subset such that $\operatorname{codim}_X(X \setminus X_0) \geq 2$, and $\mathcal{L}|_{X_0}$ is locally free, then $j_*(\mathcal{L}|_{X_0}) = \mathcal{L}$, where $j : X_0 \to X$ is the inclusion.*

Let us then propose our definition of a divisorial sheaf, which should capture both Weil divisors and Cartier divisors.

Definition 3.2. *A reflexive sheaf of rank 1 is called a divisorial sheaf.*

Trivially, any invertible sheaf is divisorial, as a locally free sheaf is reflexive, and invertible sheaves (by definition) are rank one. To see that the notion of Weil divisor is also captured by this definition, we have the following.

Theorem 3.6. *On a normal variety X, define a map Φ which assigns to a divisor $D \in Div(X)$ a divisorial sheaf given by the associated sheaf to the presheaf defined by*

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in \mathcal{K}(U)^\times \mid div(f) + D \geq 0\}.$$

Then this map is an isomorphism.

---

[6]Not to be confused with the *derived dual*, which will arise when we discuss derived categories.

First note that the sheaf $\mathcal{O}_X(D)$ defined above is a $\mathcal{O}_X$-submodule of the quotient sheaf $\mathcal{K}$, and so is automatically torsion free. Moreover we stress that this is a sheaf constructed in terms of *Weil divisors*. An example of a divisorial sheaf which is not invertible is the following.

Example:

Divisorial sheaves need not be invertible, for example, consider the affine cone $xy + z^2 = 0$. Take the closed subset defined by $V(y, z)$ (this is a line on the cone through the singular point). Since this is not principal, it cannot give an invertible sheaf, however it does define a divisorial sheaf in the same way as the theorem above states.

Note that in the case of a locally factorial condition divisorial and invertible coincide, see [22].

Proof. We follow the proof of [25], Theorem 5.2.7. Note that outside D, $\mathcal{O}_X \cong \mathcal{O}_X(D)$, and so these sheaves have rank 1. Take an open subset X_0 with complement in X of codimension at least 2, with $\mathcal{O}_X(D)|_{X_0}$ invertible. If $j : X_0 \to X$ is the inclusion, then if we can show $j_*(\mathcal{O}_X(D)|_{X_0}) = \mathcal{O}_X(D)$, then we are done by Proposition 3.5. Since Weil divisors are determined on open dense subsets with complements of codimension at least two, we can see that prime Weil divisors on $X \setminus X_0$ and $U \setminus X_0$ correspond bijectively. This means (quite literally) that $\mathcal{O}_X(D)(U) = j_*(\mathcal{O}_X(D)(U)|_{X_0})$, as required.

Conversely, if we have a divisorial sheaf $\mathcal{L}$, then there an open dense subset X_0 with codimension at least two such that $\mathcal{L}|_{X_0}$ is invertible. On this open set, simply take the Cartier divisor corresponding to this invertible sheaf, and regard it as a Weil divisor (by taking the local defining functions and sending them to prime divisors). This gives an inverse. ∎

Now that we have established the utility of divisorial sheaves, much of the same results from say, [20] follow. Indeed this even allows one to define the pullback of Weil divisors via $f^*D = f^*\mathcal{O}_X(D)$ (assuming f is surjective). One minor difference is that the product of two divisorial sheaves is given by $\mathcal{O}_X(D_1+D_2) = (\mathcal{O}_X(D_1)\otimes\mathcal{O}_X(D_2))^{\vee\vee}$ (we take the double dual to kill torsion, hence the product remains divisorial). Now any smooth variety X has a distinguished class of divisors which are intrinsic to the variety; the canonical divisors. On such a smooth variety, these are defined to be the divisors corresponding to the highest power of the sheaf of differentials, $\Omega_X^{\dim X} = \omega_X$, which is invertible as X was assumed smooth. On a singular variety there is a corresponding notion of a dualizing complex, as developed by Grothendieck, but this is not quite the same object. When X is simply normal, we can make an elementary definition for what the canonical sheaf should be.

Definition 3.3. *Let X be a normal variety and let $j : U \to X$ be the inclusion of the smooth locus (which has complement at least codimension 2). Then we can form the canonical (invertible) sheaf $\omega_{U/k} = \Omega_{U/\mathbb{C}}^{\dim U}$ on U, and then pushforward along j to give $\omega_X = j_*\omega_U$. This is referred to as the canonical sheaf on X.*

This is trivially divisorial as it satisfies the second criteria in Proposition 3.5. This then defines a Weil divisor class K_X, which we call the canonical class.

Definition 3.4. *Let X be a normal variety. We say X has canonical singularities if it satisfies the following two conditions:*

(1) *some power of the Weil divisor K_X is Cartier (that is, $\omega_X^{[r]} = (\omega_X^{\otimes r})^{\vee\vee}$ is Cartier for some r) in some neighborhood of the singularity;*
(2) *if $f : Y \to X$ is a resolution of singularities, and $\{E_i\}$ the exceptional prime divisors, then*
$$rK_Y = f^*(rK_Y) + \sum a_i E_i$$
with $a_i \geq 0$.

The minimal such r so that K_X is Cartier is called the index of the singularity, and the Weil $\mathbb{Q}$-divisor $\Delta = (1/r)\sum a_i E_i$ which satisfies
$$K_Y = f^* K_X + \Delta$$
is called the discrepancy of f.

Definition 3.5. *If $\Delta = 0$, the resolution f is called a crepant resolution.*

Actually, the second condition in the definition of canonical singularities goes by a different name, an r-Gorenstein singularity (where r is the index). A singularity which is 1-Gorenstein and Cohen-Macaulay is called a Gorenstein singularity, as this is equivalent to the local ring at that point being a Gorenstein ring. This story is a long one, and we don't have the time to discuss it here, but the equivalence of this fact is essentially [34] Theorem 18.1, or [12] Proposition 21.5. Recall that a ring R is said to be Gorenstein if it has finite injective dimension as a module over itself. That is, any injective resolution is bounded by a finite number, that number is said to be the injective dimension. The upshot is that a normal scheme X is Gorenstein if and only if K_X is Cartier [30].

Now it is a general fact that the ring of invariants of a regular ring for a "reasonable"[7] group is Cohen-Macaulay [24], in fact if the field we are working over is characteristic zero, then in fact the ring of invariants is actually also Gorenstein (see [4] and references therein). In summary:

Theorem 3.7. *Let G be a finite subgroup of $SL(2, \mathbb{C})$. Then the ring of invariants $\mathbb{C}[x, y]^G$ is a Gorenstein ring.*

For the entertainment of the reader, we also include the following, which roughly is the content of the classical McKay correspondence. For more characterizations see the source, and also [10].

Theorem 3.8 (Theorem 7.5.1, [25])**.** *Let X be a normal two dimensional variety with $x \in X$ a singular point. Then the following are equivalent:*

(1) *x is a quotient singularity and a Gorenstein singularity;*
(2) *Let $f : Y \to X$ be the minimal resolution, then $f^* K_X = K_Y$;*

[7]The actual condition is linearly reductive, which is certainly satisfied in the case of a finite group.

(3) *The exceptional divisor of the minimal resolution consists of several copies of* $\mathbb{P}^1$*, each with self-intersection* -2 *with normal crossings. The intersection graph is one of the* A_n*,* D_n*,* E_6*,* E_7*,* E_8 *Dynkin graphs.*
(4) X *is a hypersurface of* $\mathbb{C}^3$ *and the defining equation is one of those appearing in Figure 1.*

3.3. **Symplectic Algebraic Geometry.** Given a smooth algebraic variety X over a field, we can form its canonical sheaf Ω^1_X as the sheaf of "algebraic" differential 1-forms on X. A symplectic structure on X is a nonzero global section ω of $\Omega^2_X = \bigwedge^2 \Omega^1_X$ which is nondegenerate, i.e., the natural map $\Omega^1_X \to (\Omega^1 X)^\vee$ is bijective. On stalks this gives a nondegenerate 2-form $\omega_x \in \Omega^2_{X,x} = \Lambda^2(\mathfrak{m}_x/\mathfrak{m}_x^2)$. In particular, this implies that the dimension of X is even (as in the smooth case), and the highest power of ω gives a nowhere vanishing section of $\Omega_X^{\dim X} = \omega_X$, so the canonical bundle is trivial.

Example:
The first and only example that we need of such a structure is given by $\mathbb{C}^2$, with the 2-forms being given by

$$\omega = dx \wedge dy,$$

which is also a top form for the canonical sheaf on $\mathbb{C}^2$. Hence $\omega_{\mathbb{C}^2} \cong \mathcal{O}_{\mathbb{C}^2}$.

Theorem 3.9 (Theorem 6.7, [26] and references therein)**.** $\mathrm{Hilb}^n \mathbb{C}^2$ *admits a symplectic structure.*

If X is a symplectic algebraic variety, we say that a finite group G acts symplectically if $g^*(\omega) = \omega$ for all $g \in G$. We then have the following key result.

Lemma 3.9.1. *Let* G *be a finite group acting symplectically on* (X, ω)*. Then* X^G *is a symplectic subvariety.*

Proof. We skip the proof that the fixed locus is smooth, see Proposition 3.3 and the discussion afterwards. Let $x \in X^G$, and by Maschke's Theorem, the linear representation of G on T_xX splits into $T_xF \oplus N$, where F is the tangent space of the fixed locus, i.e., $T_xF = (T_xX)^G$. The form ω on X defines a morphism $T_xF \to (T_xX)^* \to N^*$, and since G preserves ω, ω_x is a morphism of represenations of G. Since N^* does not contain any trivial summands, the map is trivial by Schur's Lemma. Thus $T_xF \to (T_xF)^*$ is an isomorphism. ■

We are now in a position to prove Theorem 3.5:

Proof of Theorem 3.5. We can see from the previous two results that $X = \mathrm{Hilb}^G_{reg} \mathbb{C}^2$ acquires a symplectic structure from $\mathbb{C}^2$, as G, being a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, acts on the form $\omega = dx \wedge dy$ on $\mathbb{C}^2$ as $g^*(dx \wedge dy) = \det g(dx \wedge dy) = dx \wedge dy$.

Now the Hilbert-Chow morphism $\pi : \mathrm{Hilb}^n \mathbb{C}^2 \to S^n\mathbb{C}^2$ is proper, as it is the restriction of the projective morphism $\mathrm{Hilb}^n \mathbb{P}^2 \to S^n\mathbb{P}^2$. Hence we get the induced G-Hilbert-Chow morphism $\pi_G : X \to (S^{|G|}\mathbb{C}^2)^G \cong \mathbb{C}^2/G$ by Proposition 3.4. Away from the singular point, any point is a G-orbit of a nonzero point in $\mathbb{C}^2$, which is a reduced zero-dimensional subscheme invariant under G. Hence this establishes as isomorphism on a dense open set, and so X and $\mathbb{C}^2/G$ are birational. Since the former is nonsingular, this is a resolution of singularities.

To show this resolution is minimal, note the symplectic structure on X induces $\mathcal{O}_X = \omega_X$, and since the trivializing section of $\omega_{\mathbb{C}^2}$ is invariant under the action of G, it passes to a necessarily trivializing section of $\omega_{\mathbb{C}^2/G}$. Hence $\omega_{\mathbb{C}^2/G} \cong \mathcal{O}_{\mathbb{C}^2/G}$. Now since $\pi_G^*(\mathcal{O}_{\mathbb{C}^2/G}) = \mathcal{O}_X$ always holds, we have a crepant resolution. Note also that this implies by adjunction on X that any connected component of the exceptional divisor has self-intersection -2 (as the canonical class is trivial), and by Castelnuovo's criterion, this resolution must be minimal. ■

There are two important lessons to learn here. First, the above result is the first (and only) time that we explicitly needed the assumption that G was a subgroup of $\mathrm{SL}(2,\mathbb{C})$. This may give one the feeling that there *should* be some generalization of the whole thing. Indeed I feel the same way, but the situation seems to be complicated.

Second, since minimal resolutions are unique up to isomorphism, this is one way of seeing that $\mathrm{Hilb}^G_{reg}\,\mathbb{C}^2$ is isomorphic to a suitable blow-up of the surface $\mathbb{C}^2/G$ (as one can also obtain a minimal resolution via blowing-up. See the earlier result of Lipman). Since the blow-up of a quasi-projective (projective) variety is also a quasi-projective (projective) variety, we also obtain that $\mathrm{Hilb}^G_{reg}\,\mathbb{C}^2$ is a quasi-projective variety. This is far from true in general, but in this remarkable case it happens to hold.

4. Derived Categories

Homological algebra is perhaps one of the most important mathematical developments of the past century, and has extended its reach far into many subfields of algebra, topology, geometry, and even into more distant fields like analysis and beyond. The fundamental definition is that of a derived functor, pioneered by Grothendieck in his Tôhoku paper [16], which takes in a resolution of the object and returns an infinite sequence of groups. In some sense this is an axiomatization of the idea of homological algebra, and the notion and existence of derived functors are the fundamental objects from this viewpoint.

However a lingering questions remains: what kind of resolution should we take? Indeed there are resolutions by injectives and projectives, flat objects, free objects, etc. and a remarkable and deep fact is that whenever two such resolutions are both applicable, they give the same result for the derived functors. Grothendieck (and his student Verdier, who developed most the theory below at Grothendieck's direction) saw this as evidence that the notion of a derived functor was not actually the central idea. Rather, the resolution should be the object of focus. Following this line of though, it becomes desirable to pursue a setting where the various resolutions of an object can be though of as isomorphic, which as we are sure the reader is aware, a strictly stronger statement then saying that the (co)homology agrees.

Moreover there are good reasons why one would prefer to works with resolutions rather then with the (co)homology directly. Indeed we know from algebraic topology that there are topological spaces where all (co)homology groups agree, yet the spaces are not homeomorphic. For an example one can consider the Poincaré homology sphere as a example. On the other hand, it is a theorem of Whitehead that the corresponding complex associated to the (co)homology actually detects the difference. This would again suggest that the resolution or complex of objects is more fundamental then the (co)homology. The natural place this leads is the derived category of an abelian category. Originally it was developed as a tool for working with sheaves on a scheme, and provided the natural place to formulate Grothendieck's duality theory, which we will briefly discuss later. Since then, they have become an important tool in their own right as an invariant of schemes, and many current results hint that the derived categories of a scheme contain far more data then we had previously thought. For some examples of some of these results, see [23].

As a motivating example, let X and Y be varieties and $f : X \to Y$ a morphism. If $\mathcal{F}$ is a quasi-coherent sheaf on X, choose an injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots ,$$

which we know exists as the category of quasi-coherent sheaves has enough injectives. Applying f_* and taking cohomology, we get the higher direct images $R^i f_* \mathcal{F}$. If we did this again to another morphism, say $g : Y \to Z$, and took high direct images, there is a spectral sequence to compare the composition, i.e. $R^q g_* R^p f_* \implies R^{p+q}(g \circ f)_*$. However in the derived category, derived functors return a complex of objects (as opposed to $R^i f_*$, which return a single object), and a simple consequence of the definition gives that $\mathbf{R}g_* \circ \mathbf{R}f_* = \mathbf{R}(g \circ f)_*$. This is just one of the many "miracle formulas" which hold in the derived category but not in the ordinary setting of homological algebra.

Lets again look at an injective resolution of a sheaf $\mathcal{F}$ on a variety: $0 \to \mathcal{F} \xrightarrow{\epsilon} \mathcal{I}^\bullet$. We can rephrase this in terms of a chain map of complexes, given by

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{I}^0 & \longrightarrow & \mathcal{I}^1 & \longrightarrow & \mathcal{I}^2 & \longrightarrow & \cdots
\end{array}$$

so then regarding $\mathcal{F}$ as a complex of sheaves, concentrated solely in degree zero, and $\mathcal{I}^\bullet$ as an abstract complex not associated to $\mathcal{F}$ which is bounded below, we see that the map $\mathcal{F} \xrightarrow{\epsilon} \mathcal{I}^\bullet$ is a chain map. By definition this map is a quasi-isomorphism, i.e. a chain map which induces an isomorphism on cohomology. Then what we desire is a category where $\mathcal{F} \cong \mathcal{I}^\bullet$, for some appropriate notion of isomorphism.

Definition 4.1. *Let $\mathcal{A}$ be an abelian category and $C(\mathcal{A})$ its category of cochain complexes (where objects are cochains of objects in $\mathcal{A}$ and morphisms are chain maps). We say a morphism of chain maps is a quasi-isomorphism if it induces an isomorphism on cohomology. A morphism $f : A \to B$ is homotopic to zero if there is a chain homotopy, i.e. a degree -1 morphism $h : A \to B$, such that $f = d_B h + h d_A$.*

We denote by $C^*(\mathcal{A})$, where $* = \{+, -, b\}$, the bounded below, bounded above, and bounded versions of this category respectively. It is an easy fact that $C^*(\mathcal{A})$ is also an abelian category, and we define the cohomology of a complex to be the ordinary definition (i.e., kernel divided by image). This category is perfectly fine, and indeed much of homological algebra can be done in this setting, but for our purposes we want to first pass to a slightly different category.

Definition 4.2. *The homotopy category $K^*(\mathcal{A})$ has the same objects as $C^*(\mathcal{A})$ (where $* = \{+, -, b\}$), but morphisms are homotopy equivalence classes of chain maps.*

A priori, is is not obvious at all that this category is well-defined. However recall that $\operatorname{Hom}_{C^*(\mathcal{A})}(A, B)$ (we omit $\bullet$ signs on objects, but they should always be interpreted as complexes) is an abelian group, and it is not hard to see that the subset $\operatorname{Hom}^0_{C^*(\mathcal{A})}(A, B)$ of chain maps homotopic to zero is a subgroup. Then we can just define $\operatorname{Hom}_{K^*(\mathcal{A})}(A, B) = \operatorname{Hom}_{C^*(\mathcal{A})}(A, B)/\operatorname{Hom}^0_{C^*(\mathcal{A})}(A, B)$. Note further that since homotopic maps induce the same maps on cohomology, the functor which gives cohomology descends identically to a well-defined functor on $K^*(\mathcal{A})$. Hence the notion of a quasi-isomorphism makes sense here. Some notational changes: we will occasionally abbreviate quasi-isomorphism by "qis".

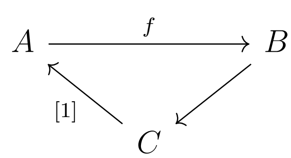
While the homotopy category $K^*(\mathcal{A})$ is not abelian, it carries a related structure, namely that of a triangulated category. The central thrust of this idea is two replace exact sequences by so-called *exact triangles*. To define these, first note that there is an endofunctor $(-)[1]$ on $K^*(\mathcal{A})$, known as the shift functor, defined as follows: given some object A, $A[1]$ is the complex obtained by shifting the indices up by one and changing the sign of the differential. That is, $A[1]^i = A^{i+1}$ and $d_{A[1]} = -d_A$. The an exact triangle (sometimes exact is omitted) is a sequence of objects and morphisms of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1],$$

and a morphism of triangles is a triple of morphisms u, v, w such that

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{g'} & A'[1] \end{array}$$

commutes. In the case where u, v, w are isomorphisms, this is called an isomorphism of triangles. Diagrammatically triangles are also represented as



and can be visualized as a "spiral staircase" where $C \to A[1]$ is a "step" upwards.

If we forget that we are dealing with the homotopy category briefly, and let $(-)[1]$ represent any additive autoequivalence, we can state the full definition of arbitrary triangulated category.

Definition 4.3. *Let $\mathcal{K}$ be an additive category with an additive autoequivalence $(-)[1]$. By a triangle in $\mathcal{K}$ we mean a sequence*

$$X \to Y \to Z \to X[1].$$

Such an additive category $\mathcal{K}$ is called a triangulated category if it is in addition equipped with a family of distinguished triangles, satisfying

(1) (a) $X \xrightarrow{id} X \to 0 \to X[1]$ *is distinguished.*
  (b) *Any morphism $f : X \to Y$ can be extended to a distinguished triangle.*
  (c) *Any triangle isomorphic to a distinguished triangle is distinguished.*

(2) *The triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished if and only if $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is distinguished.*

(3) *If, in the diagram below:*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \exists} & & \downarrow{\scriptstyle u[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

the horizontal rows are exact and the left square commutes, then there exists an arrow $Z \to Z'$ making the entire diagram commute.

(4) *"The Octahedral Axiom". We don't use it here and so we won't discuss it, but the statement can be found in say [23] or [46]. Intuitively speaking, it says roughly that if one writes a distinguished triangle as $A \to B \to C \to T(A)$ by imposing that $C = B/A$, then f defines a morphism $B/A \to C/A$ with cokernel C/B.*

Here are some formal properties:

Proposition 4.1. *Let $\mathcal{K}$ be a triangulated category and $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ a triangle. Then any composition of two consecutive morphisms are zero.*

Proof. Since we are free to "rotate" any triangle, it is enough to show that $g \circ f = 0$. Further, since shifting is an autoequivalence, it is also enoug to show that $(g \circ f)[i] = g[i] \circ f[i] = 0$ for some $i \in \mathbb{Z}$. We will show it for $i = 1$. As such consider the morphism of triangles

$$\begin{array}{ccccccc} B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] & \xrightarrow{-f[1]} & B[1] \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle g[1]} \\ C & \xrightarrow{\mathrm{id}} & C & \xrightarrow{0} & 0 & \xrightarrow{0} & C[1] \end{array}$$

where we can fill in the missing arrow by a morphism $A[1] \to 0$ making the diagram commute (by the axioms). Hence in the right-most commutative square, we have $(g \circ f)[1] = 0$, which is the claim. ■

Proposition 4.2. *Let $\mathcal{K}$ be a triangulated category and $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ a triangle. Then for any object T there is a long exact sequence of abelian groups*

$$\cdots \to \mathrm{Hom}(T, A[i]) \to \mathrm{Hom}(T, B[i]) \to \mathrm{Hom}(T, C[i]) \to \mathrm{Hom}(T, A[i+1]) \to \cdots.$$

Proof. Again by rotation, its enough to show that

$$\mathrm{Hom}(T, A[i]) \to \mathrm{Hom}(T, B[i]) \to \mathrm{Hom}(T, C[i])$$

is an exact sequence. By Proposition 4.1, we have $g[i] \circ f[i] = 0$, and hence the induced morphisms (not to be confused with the pushforward functor later on) $g_*[i] \circ f_*[i] : \mathrm{Hom}(T, A[i]) \to \mathrm{Hom}(T, C[i])$ is zero also. Hence the triple above is a complex, but now we must show its exact.

Take u in the kernel of $g_*[i]$. Consider the following diagram of triangles

$$\begin{array}{ccccccc} T[-i] & \xrightarrow{0} & 0 & \xrightarrow{0} & T[-i+1] & \xrightarrow{-\mathrm{id}} & T[-i+1] \\ \downarrow{\scriptstyle u[-i]} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle u[-i+1]} \\ B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] & \xrightarrow{-f[1]} & B[1] \end{array}$$

and by assumption on u, the left square is commutative. Hence we can fill in the remaining slot with a morphism $v : T[-i+1] \to A[1]$ making the diagram commute, but this is exactly what we needed to show. ■

Further, there is a notion of a "triangulated 5-lemma" and a notion of "split triangle" (as in a split exact sequence). We leave these straightforward generalizations to the reader. Alternatively, any books which discusses triangulated categories at any length will have explicit statements (for example, [23]).

One may wonder what the relationship between abelian and triangulated categories is in general. Recall that an abelian category is said to be semisimple if all short exact sequences split. One example of this is of course $\mathbb{C}[G]$-Mod if G is a finite group.

Theorem 4.1. *Let $\mathcal{K}$ be a triangulated category which is also abelian. Then $\mathcal{K}$ is semisimple.*

While we won't discuss generalities any further, the proof of the above is surprisingly simple once one has the notion of a split triangle.

Given all of this, we now want to expand on our remark that the homotopy category $K^*(\mathcal{A})$ is a triangulated category. For this purpose an important construction will be that of the cone of a morphism. Recall that given a morphism $f : A \to B$ between two objects, one can construct a complex called the cone of f as $C(f) = A[1] \oplus B$, with differential given by

$$d_{C(f)} = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}.$$

We then claim that complexes isomorphic to complexes of the form

$$A \to B \to C(f) \to A[1]$$

will be the exact triangles in our triangulated structure. There is an obvious short exact sequence (in $\mathcal{A}$) $0 \to B \to C(f) \to A[1] \to 0$, which induces a long exact sequence in cohomology, where the boundary map $H^{i+1}(A) = H^i(A[1]) \to H^{i+1}(B)$ is exactly f_* (the induced map on cohomology). This shows that $C(f)$ is acyclic if and only if f is a quasi-isomorphism. Further, this sequence splits if and only if f is homotopic to zero (this is actually a more general fact).

Theorem 4.2. *Let $\mathcal{A}$ be an abelian category. Then $K(\mathcal{A})$ is triangulated.*

We wont prove this, since at the very least, we have not given the full definition of a triangulated category. But we can say some thing about the other axioms. Clearly the shift functor gives an autoequivalence, and clearly $K(\mathcal{A})$ is an additive category. Note that 1(b) and 1(c) in the axioms are trivial, using the mapping cone and our definition of the triangles respectively. Now let A be any complex. The definition of $C(\mathrm{id})$ tells us that that

$$d^n_{C(f)} = \begin{pmatrix} -d^{n+1}_A & 0 \\ \mathrm{id} & d^n_A \end{pmatrix} : A_{n+1} \oplus A_n \to A_{n+2} \oplus A_{n+1}$$

is homotopic to zero via the map

$$h_n = \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}.$$

This shows axiom 1(a).

For axioms 2, one needs to show that for a triangle

$$A \xrightarrow{f} B \xrightarrow{g} C(f) \xrightarrow{h} A[1],$$

the rotated complex

$$B \xrightarrow{g} C(f) \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is actually isomorphic to

$$B \to C(f) \to C(g) \to B[1].$$

We will not say anything about the other axioms. A complete proof can be found in [46].

The inspiration for these specific triangles comes from the fact that if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence, then the map $C(f) \to C$ by first projecting to B, then applying g is a quasi-isomorphism. This naturally leads one to consider the sequence

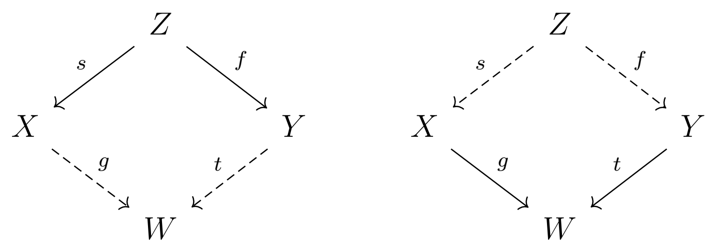
$$A \xrightarrow{f} B \to C(f) \to A[1]$$

where the maps to and from the cone are inclusion and projection respectively. We saw already that $\mathrm{Hom}(T, -)$ applied to the triangle above we get a corresponding long exact (in $\mathcal{A}$) sequence of cohomology. Emulating this fact, any functor which takes the triangle above to a long exact sequence in any abelian category is said to be a (contravariant or covariant) *cohomological* functor.

The essential reason that we have followed this path so far is that of structure. The goal is of course to find a category with the same objects as $C^*(\mathcal{A})$, yet the quasi-isomorphisms are actually isomorphisms. The process of inverting a class of maps is known as localization and in general it can be non-trivial. While there is always a large category such that any class of morphisms is invertible, its not true that this category has a good description, and even in well-behaved cases, it can destroy the structure of the original category. If however we enforce the condition that our class of morphisms which we invert satisfy some reasonable criteria, then it is plausible (and indeed possible) that the localized category have some useful structure. The solution in our case is to give up some of the abelian structure to obtain a triangulated category (this was passing to $K^*(\mathcal{A})$), then inverting the triangulated category in such a way that the resulting category is also triangulated. This was done by Verdier, and as such has the name "Verdier Localization". The criteria that we enforce on our morphisms roughly mimics the notion of a multiplicatively closed subset for the standard commutative algebra process of localizing rings.

Theorem 4.3. *The family of quasi-isomorphisms is a localizing class in the homotopy category $K^*(\mathcal{A})$, i.e. if we denote by S the class of quasi-isomorphisms, then*

(1) *S is closed under composition and contains the identities;*
(2) *for any quasi-isomorphism $s : Z \to X$ and a morphism $f : Z \to Y$, there exists $g : X \to W$ and a quasi-isomorphism $t : Y \to W$ such that the diagram (and the diagram with arrows reversed) below commute:*



(3) *for any $f, g : A \to B$, the existence of a quasi-isomorphism s such that $s \circ f = s \circ g$ is equivalent to the existence of a quasi-isomorphism t such that $f \circ t = g \circ t$.*

These axioms are chosen so that when we invert quasi-isomorphisms, the process of manipulating morphisms mimics that of fraction arithmetic. Indeed if s is a quasi-isomorphism, and its "inverse" is s^{-1}, then we can denote the composition $f \circ s^{-1}$ as f/s. Then the first condition is simply saying that the unit is invertible and that the product of invertible elements is invertible (this is the condition which is analogous to multiplicatively closed). The second is a statement about common denominators, and the third about the equality of "left" and "right" fractions. A key point here is that quasi-isomorphisms form a localizing class in the homotopy category, but *not* in the category of chain complexes (another reason why we pass to $K^*(\mathcal{A})$ first). That this gives a triangulated category, and agrees with the construction we loosely sketched above is due to Verdier. Then we can define the derived category as the following:

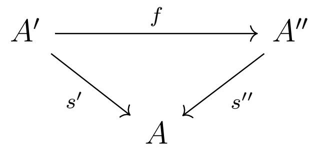
Theorem 4.4 ([45])**.** *Let $\mathcal{A}$ be an abelian category and $K^*(\mathcal{A})$ is homotopy category. Then there is a category $\mathrm{D}^*(\mathcal{A})$, the derived category of $\mathcal{A}$, and a functor $Q : K^*(\mathcal{A}) \to \mathrm{D}^*(\mathcal{A})$ such that*

(1) *if f is a quasi-isomorphism, then $Q(f)$ is an isomorphism;*
(2) *if $F : K(\mathcal{A}) \to C$ is any functor satisfying the first condition, then there is a unique functor $G : \mathrm{D}(\mathcal{A}) \to C$ such that $F = G \circ Q$.*

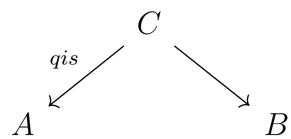
Moreover the derived category $\mathrm{D}^*(\mathcal{A})$, where $* = \{+, -, b\}$, is the category with objects the same as $K^*(\mathcal{A})$ (the same as $C^*(\mathcal{A})$), and morphisms are defined as

$$\mathrm{Hom}_{\mathrm{D}^*(\mathcal{A})}(A, B) = \varinjlim_{I_A} \mathrm{Hom}_{K^*(\mathcal{A})}(A', B),$$

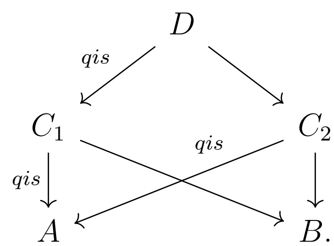
where I_A is the category whose objects are quasi-isomorphisms $s : A' \to A$ and morphisms are commutative diagrams



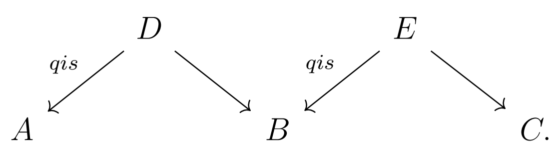
As we see, the existence and a universal property of the derived category is given above. However the definition of the morphisms in this category is perhaps a bit opaque. All it is really saying is that the morphisms are equivalence classes of "roofs":



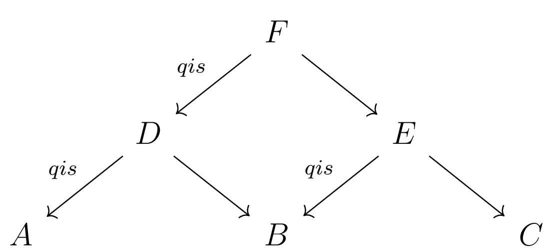
where the left map is a quasi-isomorphism and the right map is an arbitrary homotopy equivalence class of chain maps. Two such roofs are considered equivalent if there is a larger roof above both of them. Explicitly:



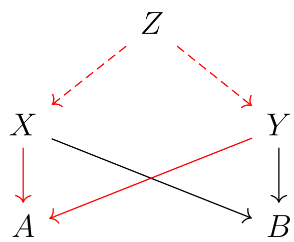
To show completely that this is a category we should show that there is an associate composition law. While we leave the associativity to the reader (its similar), we will instead show that the category is additive. Take two roofs $A \to B$ and $B \to C$. We can then form the diagram:



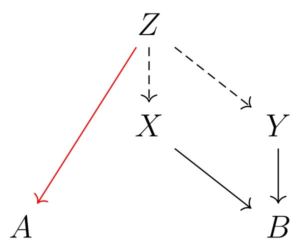
It is then an axiom of the localizing class that we can "fill in" the middle to form a larger roof, with the left map being a quasi-isomorphism, i.e.



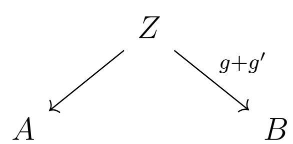
and so the composition of the two maps is defined as the larger roof. To add morphisms, take two maps in the derived category between A and B



where we have "filled-in" the roof corresponding to the two quasi-isomorphisms (the solid red arrows) with two more quasi-isomorphisms (the dashed red arrows). By the axioms for a localizing class (Theorem 4.3), this square is commutative, and hence we can replace it by a single morphism $Z \to A$. Now we have the diagram:



where the red arrow comes from the commutative square, and two morphisms $Z \to B$. Denoting the morphisms (which factor through X and Y respectively) by g, g', we can add them since they are genuine morphisms in $K(\mathcal{A})$. Hence the sum is defined to be



It remains to see that quasi-isomorphisms are invertible, however this is easier then the additivity of the category, and so we implore the reader to check this fact themselves, as it's an excellent exercise to get familiar with the techniques we exhibited above. While you're at it, you may as well go ahead and check that cohomology is well-defined on this category (hint: quasi-isomorphisms induce isomorphisms on cohomology). We can also exhibit the triangulated structure on $\mathrm{D}^*(\mathcal{A})$. This is defined similarly to the homotopy category: given some object A, we define $A[1]$ the same, and for a morphism represented by a roof (s, f), the shift is the roof represented by $(s[1], f[1])$. Then a triangles in $\mathrm{D}^*(\mathcal{A})$ are simply the image of triangles in $K^*(\mathcal{A})$ under the image of the localization functor.

4.1. **Derived Functors.** Let $F : \mathcal{A} \to \mathcal{B}$ be a functor of abelian categories. If F is not exact, then it does not carry acyclic objects to acyclic objects, and hence the extension to a functor $\mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ does not make sense. If F does preserve acyclic objects, then still it is not obvious that it extends. However, we have the following.

Proposition 4.3 (Lemma 2.44, [23])**.** *A functor $F : K^*(\mathcal{A}) \to K^*(\mathcal{B})$ induces a functor $\mathrm{D}^*(\mathcal{A}) \to \mathrm{D}^*(\mathcal{B})$ whenever at least one of the following is true:*

(1) *under F a quasi-isomorphism is mapped to a quasi-isomorphism, or*
(2) *the image of an acyclic complex is again acyclic.*

However there are many situations where we would want a functor between derived categories which may be associated to some functor which satisfies neither. As an illustration, if $f : X \to Y$ is a morphism of varieties, we would want a "derived" pushforward $\mathbf{R}f_*$. It turns out that there is such a functor, which can be understood as an approximation to f_*, or perhaps more accurately as an extension of f_*. As above let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, and assume that we have enough injectives in both. It is a fact that there is a natural equivalence $K^+(\mathrm{Inj}(\mathcal{A})) \to \mathrm{D}^+(\mathrm{Qcoh}\, X)$ induced by the functor $Q : K^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{A})$ that this is essentially surjective is obvious, since the category admits enough injectives, every object is quasi-isomorphic to a complex of injective objects (there is some subtlety here, as we should be working with complexes in $K^+(\mathcal{A})$, not in $\mathcal{A}$ itself, but let us not worry about this). Fully faithful requires some elementary facts about homological algebra which we will not repeat here, see Proposition 2.40, [23] for all of the details. Choose a quasi-inverse ι, given by choosing a complex of injective objects quasi-isomorphic to a bounded below complex in $\mathrm{D}^+(\mathcal{A})$, and let $K(F)$ be the functor $K^+(\mathcal{A}) \to K^+(\mathcal{B})$ induced by F. Given an object in $\mathrm{D}^+(\mathcal{A})$, we choose a quasi-inverse to the equivalence earlier, resulting in a complex of injectives in $K^+(\mathrm{Inj}(\mathcal{A}))$, and then apply the functor $K(F)$ to the chosen injective resolution. Since this takes injectives to injectives, this induces a functor to the derived category by applying $Q_{\mathcal{B}}$. Diagrammatically:

$$\begin{array}{ccc} \mathrm{D}^+(\mathrm{Qcoh}\ X) & \xrightarrow{\mathbf{R}F} & \mathrm{D}^+(\mathrm{Qcoh}\ Y) \\ \cong\uparrow\ \downarrow\iota & \nearrow_{Q_{\mathcal{B}}\circ K(F)} & \\ K^+(\mathrm{Inj}(\mathrm{Qcoh(X)})) & & \end{array}$$

We denote the composition by $\mathbf{R}F$, and call the result the right derived functor of F. Explicitly, if we wanted to compute this functor on some object A in $\mathrm{D}^+(\mathcal{A})$, we would choose an injective resolution I, then apply $K(F)(I)$, followed by the localization functor. However since the localization functor does not alter objects, the image of this functor is just $K(F)(I)$. Further, given some complex A in the derived category, we see that almost by definition, the classical higher derived functors F $R^iF : \mathcal{A} \to \mathcal{B}$ can be described as $R^iF(A) = H^i(\mathbf{R}F(A))$. Analogously, we can (if we assume the existence of enough projectives) also give the construction of the left derived functor of a right exact functor of abelian categories.

However we have so far assumed that our categories have enough injectives/projectives, which is in general far from true. To remedy this we will need the more general notion in which derived functors can be constructed, that of an *adapted class* (see [21]).

Definition 4.4. *Let F be a left (right) exact functor. A class of objects $\mathcal{K} \subset K^*(\mathcal{A})$ is called a left (right) adapted class for F if the following hold:*

(1) *F takes acyclic objects in $C^*(\mathcal{K})$ to acyclic objects,*
(2) *any object in $\mathcal{A}$ admits a momomorphism $A \to K$ (epimorphism $K \to A$), K an object in $\mathcal{K}$, and*
(3) *$\mathcal{K}$ is closed under direct sums.*

Further, the inclusion of categories $i : K^*(\mathcal{K}) \to K^*(\mathcal{A})$ *induces an equivalence of triangulated categories* $\Phi : K^*(\mathcal{K})[Qis^{-1}] \to D^*(\mathcal{A})$*, where "Qis" is the localizing class of quasi-isomorphisms.*

Once we have such an adapted class established, the construction of the derived functors goes word for word the same as when we had enough injectives as we demonstrated above. Even further, the derived functor does not depend on the choice of adapted class once we have existence, up to functor isomorphism [21]. If our abelian category has enough injectives, then the class of injective objects forms a left adapted class for all left exact functors. This captures the classical fact that for any two applicable resolutions of of an object, the derived functors should agree.

Example:

Since we have not given an example in a while, let us now pause to discuss the example of $\mathbf{R}\operatorname{Hom}$. Assume for now that $\mathcal{A}$ has enough injectives, and consider the additive functor $F = \operatorname{Hom}_{\mathcal{A}}(A, -)$ from $\mathcal{A}$ to the category of abelian groups. It is a trivial exercise to see that this functor is left exact, but in general is not right exact (in fact, it'll be right exact for a complex of injectives, by definition). The right derived functor is then denoted by $\mathbf{R}\operatorname{Hom}(A, -) : D^+(\mathcal{A}) \to D^+(\mathrm{Ab})$, and the higher derived functors are

$$\operatorname{Ext}^i_{\mathcal{A}}(A, -) = R^i \operatorname{Hom}_{\mathcal{A}}(A, -).$$

To see all of this in more detail, we first extend the functor $\operatorname{Hom}_{\mathcal{A}}(A, -)$ to one on $K^+(\mathcal{A})$ by turning it into a complex via $K\operatorname{Hom}_{\mathcal{A}}(A, B^\bullet) = \operatorname{Hom}^\bullet(A, B^\bullet)$ (we have emphasized the fact that these are complexes, but we will continue to omit the indexing in most cases) where $\operatorname{Hom}^i(A, B^\bullet) = \operatorname{Hom}_{\mathcal{A}}(A, B^i) = \operatorname{Hom}_{K^+(\mathcal{A})}(A, B^\bullet[i])$. To extend this to a morphism of derived categories, we simply replace B with a complex of injective objects I quasi-isomorphic to X, and then apply the extended functor to get

$$\operatorname{Hom}^i_{\mathcal{A}}(A, B) = H^i(\mathbf{R}\operatorname{Hom}(A, I)) = \operatorname{Hom}_{K^+(\mathcal{A})}(A, I[i]) = \operatorname{Hom}_{\mathrm{D}^+(\mathcal{A})}(A, B[i]).$$

Note that this gives that $\operatorname{Ext}^i_{\mathcal{A}}(A, B) = \operatorname{Hom}_{\mathrm{D}^+(\mathcal{A})}(A, B[i])$ which is a nontrivial statement!

Derived categories also provide a fertile ground for spectral sequences. See for example the following result.

Theorem 4.5 (Proposition 2.66, [23])**.** *Suppose* $\mathcal{A}$ *has a left adapted class for a left exact functor* F*, and* F *maps this left adapted class to another left adapted class in* $\mathcal{B}$ *for a left exact functor* G*. Then there is a natural isomorphism of functors* $\mathbf{R}(G \circ F) \to \mathbf{R}G \circ \mathbf{R}F$*.*

The proof of this is actually trivial (maybe the reader should try), but we will postpone the proof until we are in the equivariant setting. If you actually wanted to compute the classical derived functors from the above, one would need a spectral sequence. Recall that a spectral sequence consists of the following data.

Definition 4.5. *A spectral sequence is a collection* $((E_r, d_r), H^n)$*,* $n, r \in \mathbb{Z}$*,* $r \geq 1$ *of bicomplexes* (E_r, d_r) *and a collection of objects* H^n *called the limit of the spectral* *sequence, together with a filtration*

$$\cdots \subset F^i(H^n) \subset F^{i-1}(H^n) \subset \cdots,$$

such that

(1) *each element of the complex E_r satisfies $E_r^n \cong \bigoplus_{p+q=n} E_r^{p,q}$, $p, q \in \mathbb{Z}$,*
(2) *the composition of $E_r^{p,q} \to E_r$ with d_r defines a morphism*
$$d_r^{p,q} : E_r^{p,q} \to E^{p+r,q-r+1},$$
(3) *there are isomorphisms*
$$\alpha_r^{p,q} : \mathrm{Ker}(\mathrm{Coker}(d_r^{p-r,q+r-1}) \to E_r^{p+r,q-r+1}) \cong E_{r+1}^{p,q},$$
(4) *there exists r_0 such that $d_r^{p,q}$ and $d_r^{p-r,q+r-1}$ are equal to zero for $r \geq r_0$ (we say the spectral sequence* degenerates *at r_0). The isomorphic objects E_r, $r \geq r_0$ are denoted $E_\infty^{p,q}$, and*
(5) *for each $p, q \in \mathbb{Z}$, there is an isomorphism $E_\infty^{p,q} \to \mathrm{gr}^p(H^{p+q})$, where the latter is the p-th graded part of the associated graded ring to the given filtration.*

We often write a spectral sequence in the following way: $E_r^{p,q} \implies H^n$. Note that although we have required a spectral sequence to degenerate, not all authors require this, and there are many valid examples of spectral sequences which do not degenerate. Thus to compute both sides of Theorem 4.5, one would need to show for any complex A, the existence of a spectral sequence $E_2^{p,q} = R^pG(R^qF(A)) \implies R^{p+q}(G \circ F)(A)$.

Example:

Lets consider an example from complex differential geometry, the Frölicher spectral sequence (also sometimes called the Hodge-de Rham spectral sequence). Let X be a compact complex manifold, and Ω_X^n the sheaf of holomorphic n-forms on X. It is well known that this decomposes into a direct sum of sheaves

$$\Omega_X^n \cong \Omega_X^{p,q},$$

where sections of the latter are called forms of type-p, q. Along with this decomposition comes the fact that we can write the differential $d : \Omega_X^n \to \Omega_X^{n+1}$ as $d = \partial + \overline{\partial}$, where $\partial : \Omega_X^{p,q} \to \Omega_X^{p+1,q}$ and $\overline{\partial} : \Omega_X^{p,q} \to \Omega_X^{p,q+1}$, where each is a differential.

Hence $(\Omega_X^{p,q}, (\partial, (-1)^p\overline{\partial}))$ is a double complex in the abelian category of sheaves on X, with a total complex (Ω_X^n, d). Now Dolbeaut's theorem says that $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$, i.e. that each column $\Omega_X^{\bullet,q}$ is a resolution of Ω_X^p, and each row $\Omega_X^{p,\bullet}$ is a resolution of $\overline{\Omega_X^q}$. Further, the total complex is a resolution of the constant sheaf $\mathbb{C}_X$ on X.

All together, this gives the Frölicher spectral sequence, which can be written

$$E_1^{p,q} = H^q(X, \Omega_X^p) \implies H^n(X, \mathbb{C}),$$

and is X is Kähler, the spectral sequence will degenerate on the first page, giving the famous complex Hodge decomposition

$$H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^q(X, \Omega_X^p),$$

which comes along with a fair number of other bells and whistles.

Example:

For an example which is closer to algebraic geometry, if we have a complex A with a Cartan-Eilenberg resolution[a], we get a related spectral sequence

$$E_1^{p,q} = R^q F(A^p) \implies R^{p+q}F(A).$$

Consider the example for when $F = \Gamma(X, -)$ for X a scheme, we can use this spectral sequence to compute the Hodge numbers of a complete intersection X of a quadric and cubic hypersurface in $\mathbb{P}^3$. Since this is a curve, its enough by Serre duality to compute the cohomology of the structure sheaf $\mathcal{O}_X$. We then use the Koszul resolution (if the reader has not seen Koszul resolutions, we discuss them in more detail later on when discussing the equivariant derived category of the plane):

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-5) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0,$$

and the key observation here is that the structure sheaf $\mathcal{O}_X$ is quasi isomorphic to the Koszul resolution. So we re-write it as $A \to \mathcal{O}_X$, where $A^0 = \mathcal{O}_{\mathbb{P}^3}$, $A^{-1} = \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-3)$ and $A^{-2} = \mathcal{O}_{\mathbb{P}^3}(-5)$, and all others zero.

Now if we take $F = \Gamma(X, -)$, this is a left exact functor, and hence we see that the spectral sequence above translates into $E_1^{p,q} = H^q(X, A^p) \implies H^{p+q}(X, \mathcal{O}_X)$, where we have used that $A \cong \mathcal{O}_X$ in the derived category. Then arranging the E_1 page in a diagram (where each entry is the dimension of the cohomology, entries not listed are all zero):

$$\begin{array}{c|ccc}
3 & 4 \longrightarrow & 0 \longrightarrow & 0 \\
2 & 0 \longrightarrow & 0 \longrightarrow & 0 \\
1 & 0 \longrightarrow & 0 \longrightarrow & 0 \\
0 & 0 \longrightarrow & 0 \longrightarrow & 1 \\
\hline
 & -2 & -1 & 0
\end{array}$$

we see that all arrows either start from zero or end at zero, and hence the spectral sequence stabilizes on the first page. Summing the anti-diagonals, we see that $h^0(\mathcal{O}_X) = 1$, $h^1(\mathcal{O}_X) = 4$, which agrees with the result from that a canonical curve of genus 4 is a complete intersection of a quadric and cubic in $\mathbb{P}^3$. By applying Serre duality appropriately, we get the two other Hodge numbers for this curve.

[a] A Cartan-Eilenberg resolution of a complex A is a double complex C, together with a map $A^i \to C^{i,0}$ satisfying:

(1) $C^{i,j} = 0$ for $j < 0$,
(2) the sequences $A^i \to C^{i,0} \to C^{i,1} \to \cdots$ are injective resolutions of A^i, inducing injective resolutions of $\operatorname{Ker} d_A^i$, $\operatorname{Im} d_A^i$, and $H^i(A)$, and

(3) the sequences $C^{\bullet,j}$ are split for all j.

In particular, whenever a category has enough injectives, such resolutions exist.

Now let X be a scheme, and let $\Gamma(X,-) : \mathcal{O}_X - \mathrm{Mod} \to \Gamma(X,\mathcal{O}_X) - \mathrm{Mod}$ be the global sections functor. We know that the injective sheaves are an adapted class, and so we can derive this to get the right derived functor $\mathbf{R}\Gamma(X,-)$, along with $H^n(X,\mathcal{F}) = R^n\Gamma(X,\mathcal{F}) = H^n(\mathbf{R}\Gamma(X,\mathcal{F}))$.

Example:

Let $f : X \to Y$ be a morphism of schemes. Then the composition of functors is $\Gamma(Y,-)\circ f_*(-) = \Gamma(X,-)$, and we find that the composition of derived functors, by the previous theorem, is,

$$\mathbf{R}\Gamma(Y,-)\circ \mathbf{R}f_*(-) = \mathbf{R}\Gamma(X,-).$$

The corresponding spectral sequence to compute both sides is

$$E_2^{p,q} = H^p(Y, R^q f_*\mathcal{F}) \implies H^{p+q}(X,\mathcal{F}),$$

and this is a special case of the Leray spectral sequence, which we will discuss shortly.

4.2. **Derived Categories of Sheaves.** Now let us specialize to the situation that we are interested in. X is now a smooth quasi-projective variety, and we will focus on the abelian categories of coherent and quasi-coherent sheaves. In general however we will mostly be interested in the coherent case, as $\mathrm{D}(\mathrm{Qcoh}\, X)$ is usually too big. Specifically, we would like to work with $\mathrm{D}^b(\mathrm{Coh}\, X)$, the bounded derived category of coherent sheaves; for convenience let us introduce the shorthand $\mathrm{D}^b(\mathrm{Coh}\, X) = \mathrm{D}^b(X)$. There are a few immediate problems however, namely injective objects are typically quasi-coherent[8], and we cannot take unbounded injective resolutions. For this reason we will first work with $\mathrm{D}(\mathrm{Qcoh}\, X)$, the unbounded derived category of quasi-coherent sheaves. The idea is then to "approximate" the category of coherent sheaves by a particularly nice subcategory of $\mathrm{D}(\mathrm{Qcoh}\, X)$, denoted $\mathrm{D}^b_{coh}(\mathrm{Qcoh}\, X)$ where an object is a complex of quasi-coherent sheaves, and each cohomology sheaf $\mathscr{H}^i(\mathcal{F})$ is coherent and $\mathscr{H}^i(\mathcal{F}) = 0$ for $|i| \gg 0$. There is is natural injection $\mathrm{D}^b(\mathrm{Coh}\, X) \to \mathrm{D}^b_{coh}(\mathrm{Qcoh}\, X)$, and under reasonable conditions, this is an equivalence.

Proposition 4.4 (Proposition 3.5, [23])**.** *The natural functor* $\mathrm{D}^b(X) \to \mathrm{D}^b(\mathrm{Qcoh}\, X)$ *is a fully faithful functor of triangulated categories. It defines an equivalence with the full triangulated subcategory* $\mathrm{D}^b_{coh}(\mathrm{Qcoh}\, X)$ *of bounded complexes of quasi-coherent sheaves with coherent cohomology sheaves.*

Note that the cohomology sheaves $\mathscr{H}^i(A^\bullet)$, which are the cohomology of a complex of sheaves, is not the same as the *sheaf cohomology* of a scheme X, $H^i(X, A)$.

[8]They might not even be quasi-coherent actually, and so a completely rigorous treatment would have us prove Proposition 4.4 below in the context of quasi-coherent sheaves. That is, the fully faithful inclusion $\mathrm{D}^b(\mathrm{Qcoh}\, X) \to \mathrm{D}(\mathcal{O}_X\text{-Mod})$ is an equivalence with the subcategory of complexes of $\mathcal{O}_X$ modules with bounded quasi-coherent cohomology sheaves.

Proof. The natural functor (i.e., the inclusion) is obviously full and faithful. Now given some surjective morphism $\mathcal{G} \to \mathcal{F}$ in $\mathrm{Qcoh}\, X$, with $\mathcal{F}$ coherent, it is a fact that there is a coherent subsheaf $\mathcal{G}' \subset \mathcal{G}$ such that the restriction $\mathcal{G}' \to \mathcal{F}$ is still surjective.

Let $\mathcal{G}$ be a bounded complex of quasi-coherent sheaves with coherent cohomology sheaves $\mathscr{H}^i$. Since the complex is bounded, we may assume that $\mathcal{G}^i = 0$ for i large enough. Now suppose that $\mathcal{G}^i$ is coherent for all $i > r$, for some r (certainly taking r large enough is ok, as then the sheaves are all zero, hence coherent). Then the image sheaf $\mathrm{Im}\, d^r \subset \mathcal{G}^{r+1}$ is coherent, and since d^r is surjective on its image there exists a coherent subsheaf $\mathcal{F}_1^r$ such that $d^r(\mathcal{F}_1^r) = \mathrm{Im}\, d^r$. We also get a surjection $\mathrm{Ker}\, d^r \to \mathscr{H}^r$, and by assumption, since the cohomology sheaves are coherent, there is a coherent subsheaf $\mathcal{F}_2^r$ of $\mathrm{Ker}\, d^r$ which also surjects onto $\mathscr{H}^r$.

Now replace $\mathcal{G}^r$ with the coherent sheaf $\mathcal{F}_1^r \oplus \mathcal{F}_2^r = \mathcal{F}^r$ and replace $\mathcal{G}^{r-1}$ with $(d^{r-1})^{-1}(\mathcal{F}^r)$. With this modification, we have not changed either $\mathscr{H}^r$ or $\mathscr{H}^{r+1}$, and now $\mathcal{G}^i$ are coherent for all $i > r-1$. By induction then we have constructed a complex of coherent sheaves which is quasi-isomorphic to the original complex. ■

The idea is then to derive all of our functors in the quasi-coherent setting, then use this equivalence to pass to coherent sheaves. In particular this result holds for any noetherian scheme of finite dimension, even if singular or otherwise. From now on, we identify $D^b(X)$ with $D^b_{coh}(\mathrm{Qcoh}\, X)$, as in the latter (assuming our scheme is regular and all that) we can derive our functors without worry, and we cannot in the former. In the case of a regular scheme, one can reduce even further, as any bounded complex of coherent sheaves is quasi-isomorphic to a perfect complex. Recall that a perfect complex is a bounded complex of locally free sheaves of finite rank, i.e. vector bundles. For many purposes, especially when the scheme in question is not smooth, the subcategory $\mathrm{D}_{perf}(X)$ of perfect objects (as defined above) is the preferred object to work with.

Proposition 4.5. *On an affine variety X, the category $\mathrm{Qcoh}\, X$ has both enough projectives and injectives.*

Proof. It is well known (see for example Corollary II.5.5 in [20]) that on an affine scheme $X = \mathrm{Spec}\, A$, $\mathrm{Qcoh}\, X$ is equivalent to the category A-Mod. Since this category always has enough projectives and injectives (see [46]), we are done. ■

This second result will help us only to some extent as we will later focus on the (equivariant) derived category of $\mathbb{C}^2$, but its not true that the resolution of $\mathbb{C}^2/G$ is affine. So we will still need to worry about finding appropriate adapted classes to derive all of our functors.

Direct Images (Pushforward). Let $f : X \to Y$ be a morphism of schemes, then we know that the direct image of a quasi-coherent sheaf is quasi coherent ([20]), and so if we use injective objects as an adapted class, we get a derived functor $\mathbf{R}f_* : \mathrm{D}^+(\mathrm{Qcoh}\, X) \to \mathrm{D}^+(\mathrm{Qcoh}\, Y)$. Since $H^i(\mathbf{R}f_*(\mathcal{F})) = R^i f_*(\mathcal{F})$ are the higher direct images, which are simply the associated sheaves to the cohomology sheaf, we know that they must eventually vanish for $i > \dim X$ by a result of Grothendieck. This means that $\mathbf{R}f_*$ descends to a functor $\mathbf{R}f_* : \mathrm{D}^b(\mathrm{Qcoh}\, X) \to \mathrm{D}^b(\mathrm{Qcoh}\, Y)$. If in addition f is proper, then the higher direct images of a coherent sheaf are coherent (Theorem II.8.8, [20]). So then in this case we get a functor $\mathbf{R}f_* : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$.

Another adapted class is particularly useful, namely flasque sheaves. In general the pushforward of an injective sheaf is not injective, to if we wanted to use Theorem 4.5 on a composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

we run into trouble, as the image of an injective resolution does not land in the adapted class of injectives for g_*. However it is true that the pushforward of a flasque sheaf is flasque, and its an elementary exercise (see [20]) to show that this satisfies the axioms to be an adapted class for the pushforward functor. Hence using flasque sheaves, we do get a comparision $\mathbf{R}(g \circ f)_* = \mathbf{R}g_* \circ \mathbf{R}f_*$. This leads to the so-called Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \implies H^{p+q}(X, \mathcal{F}),$$

where $\mathcal{F}$ should be regarded as a complex of sheaves. If we take $g : X \to \operatorname{Spec} \mathbb{C}$, then this reduces to the special case we saw in the previous section.

Hom. Let $\mathcal{F}$ be some quasi-coherent sheaf on our variety X. Then it is an elementary fact that the internal hom $\mathscr{H}om(\mathcal{F}, -)$ is a left exact functor on $\operatorname{Qcoh} X$. Since this category has enough injectives, the right derived internal hom exists $\mathbf{R}\mathscr{H}om(\mathcal{F}, -) : \mathrm{D}^+(\operatorname{Qcoh} X) \to \mathrm{D}^+(\operatorname{Qcoh} X)$, with, by definition,

$$R^i \mathscr{H}om(\mathcal{F}, -) = H^i(\mathbf{R}\mathscr{H}om(\mathcal{F}, -)) = \mathscr{E}xt^i(\mathcal{F}, -).$$

In particular, $\mathscr{E}xt^i(\mathcal{F}, \mathcal{E})$ is coherent if both $\mathcal{F}$ and $\mathcal{E}$ are. If the variety is regular, then the higher Ext's will vanish, and restricting to the coherent sheaves yields a functor $\mathbf{R}\mathscr{H}om(\mathcal{F}, -) : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$. Using this, we define, for a complex of sheaves $\mathcal{F}$, the *derived dual* $\mathcal{F}^\vee = \mathbf{R}\mathscr{H}om(\mathcal{F}, \mathcal{O}_X)$.

Tensor. For any two bounded complexes $\mathcal{M}$, $\mathcal{N}$ of sheaves in $C(\operatorname{Qcoh} X)$, we can define the tensor product as the complex $\mathcal{M} \otimes \mathcal{N}$ with

$$(\mathcal{M} \otimes \mathcal{N})^i = \bigoplus_{j+k=i} \mathcal{M}^j \otimes \mathcal{N}^k, \quad d^i(x^j \otimes y^k) = d_{\mathcal{M}}(x^j) \otimes y^k + (-1)^{j+k} x^j \otimes d_{\mathcal{N}}(y^k).$$

On a smooth variety every coherent sheaf $\mathcal{F}$ has a finite resolution of locally free sheaves $\mathcal{E}^\bullet \to \mathcal{F} \to 0$. Since locally free sheaves are flat, we use this to get a functor $\overset{\mathbf{L}}{\otimes} : D^b(X) \times D^b(X) \to D^b(X)$. The classical higher derived functors are by definition the Tor sheaves. Computing the derived tensor product on a finite free resolution shows that it is commutative and associative up to functorial isomorphism.

Pullback. Now the functor $f^* : \operatorname{Coh} Y \to \operatorname{Coh} X$ is given by $\mathcal{F} \mapsto f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. However it is an elementary fact that f^{-1} is already exact! Hence it is enough to simply derive the tensor product as above, and assuming that Y is regular (for our purposes, both X and Y will be regular), we can again take finite locally free resolutions, and get a functor $\mathbf{L}f^* : \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$. Further one can check (by replacing the complexes by the correct adapted classes), that there is an adjunction $\mathbf{L}f^* \dashv \mathbf{R}f_*$.

4.2.1. *Compatibilities.* There are a many compatibilities between all of the derived functors we have given so far, indeed many of these are referred to as "magic formulas" of the derived category. Of all, by far the most important ones are that of cohomological flat base change, the projection formula, and Grothendieck-Verdier duality. In this section we will simply give their statements, any proofs will be delayed until we discuss the equivariant setting (as the results there generalize these by taking the trivial group). As such this section is mainly intended as a reference, but we have tried to be clear as to how one would use these compatibilities in practice.

Proposition 4.6 (Cohomological flat base change)**.** *Consider a fiber square of algebraic varieties*

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{\bar{g}} & X \\ \downarrow{\bar{f}} & & \downarrow{f} \\ S' & \xrightarrow{g} & S \end{array}$$

Then for any complex of $\mathcal{O}_X$-modules $\mathcal{M}$, there is a natural morphism

$$\mathbf{L}g^*\mathbf{R}f_*\mathcal{M} \to \mathbf{R}\bar{f}_*\mathbf{L}\bar{g}^*\mathcal{M}.$$

Moreover if $\mathcal{M}$ has quasi-coherent cohomology sheaves and either f or g is flat, the above morphism is an isomorphism.

We wont prove this, but the proof can be found in any basic text on derived categories of sheaves. The classical statement can also be found in many basic texts. The proof of this essentially reduces to the proof that can be found in [20]. There is also a variant where f is simply smooth and proper.

Proposition 4.7 (Projection formula)**.** *Let $f : X \to Y$ be a morphism of varieties. For every complex $\mathcal{M}$ of $\mathcal{O}_X$-modules and every complex $\mathcal{N}$ of $\mathcal{O}_Y$-modules, there is a functorial morphism in $\mathrm{D}(\mathcal{O}_Y - Mod)$*

$$\mathbf{R}f_*(\mathcal{M}) \overset{\mathbf{L}}{\otimes} \mathcal{N} \to \mathbf{R}f_*(\mathcal{M} \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathcal{N})$$

which is an isomorphism if $\mathcal{N}$ has quasi-coherent cohomology.

The proof again reduces to the classical statement, the exercise can be found in [20]. Let us illustrate a scenario of use. Consider the product over $\operatorname{Spec}\mathbb{C}$ given by $X \overset{\pi_X}{\leftarrow} X \times Y \overset{\pi_Y}{\rightarrow} Y$, with structure maps s_X and s_Y respectively. Then given $\mathcal{F} \in \mathrm{D}^b(X)$ and $\mathcal{E} \in \mathrm{D}^b(Y)$, we consider the exterior tensor product $\pi_X^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \pi_Y^*\mathcal{E}$. Then we apply $\mathbf{R}\pi_{Y*}$ to the exterior tensor product, then use the projection formula and base change:

$$\mathbf{R}\pi_{Y*}(\pi_X^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \pi_Y^*\mathcal{E}) = \mathbf{R}\pi_{Y*}\pi_X^*\mathcal{F} \otimes \mathcal{E} = s_Y^*\mathbf{R}s_{X*}\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{E}.$$

Now note that the product has $s_Y \circ \pi_Y = s_X \circ \pi_X$ as its structure map, so pushing forward along the composition is the same as cohomology. Hence applying $\mathbf{R}s_{Y*}$ to the first and last expression above, and using the projection formula to simply the latter,

$$\mathbf{R}\Gamma(X \times Y, \pi_X^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \pi_Y^*\mathcal{E}) = \mathbf{R}\Gamma(X, \mathcal{F}) \otimes \mathbf{R}\Gamma(Y, \mathcal{E}),$$

which is the Künneth formula. Since the global section functor lands in $\mathrm{Vect}_{\mathbb{C}}$, the tensor product on the right is actually an ordinary tensor product.

As we mentioned earlier, there are many other compatibilities between the derived functors between derived categories of sheaves (the interested reader can consult [23] for more), but the last result we want to discuss is that of duality. Recall the classical statement of Serre duality:

Theorem 4.6 (Serre Duality, [20], Theorem 7.6)**.** *Let X be a projective scheme over an algebraically closed field k, let ω_X° be a dualizing sheaf on X, and $\mathcal{O}_X(1)$ a very ample sheaf on X, then for all $i \geq 0$ and $\mathcal{F}$ coherent on X, there are natural functorial maps*

$$\theta^i : \mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^*$$

such that the following are equivalent:

1. *X is Cohen-Macaulay and equidimensional,*
2. *for any locally free sheaf $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}(-q)) = 0$ for $i < n$ and q large enough, and*
3. *the maps θ^i are isomorphisms for all $i \geq 0$ and $\mathcal{F}$ coherent.*

Now let's look at Serre duality in the context of the derived category. For simplicity, lets assume that assume that the scheme is smooth. It obtains the particulary natural form:

$$\mathrm{Hom}_{\mathrm{D}(X)}(\mathcal{F}, \omega_X^\circ) \cong \mathrm{Hom}_{\mathrm{D}(X)}(\mathcal{O}_X, \mathcal{F})^*.$$

Indeed we know that higher derived functors of Hom are the Ext groups, so this meshes with what we've seen so far. Further, the right hand side is particularly simple, as $\mathscr{H}om(\mathcal{O}_X, -)$ is the identity functor, so taking global sections and using the local-to-global spectral sequence gives you $R^i \mathrm{Hom}(\mathcal{O}_X, \mathcal{F}) = H^i(X, \mathcal{F})$ for a single sheaf concentrated in a degree zero.

The left hand side can also be simplified, via the derived version of Hom-$\otimes$ adjunction. This turns out to be $\mathbf{R}\,\mathrm{Hom}_{\mathrm{D}(X)}(\mathcal{O}_X, \mathcal{F}^\vee \overset{L}{\otimes} \omega_X^\circ)$. Now since the scheme is smooth, the dualising sheaf is simply the canonical sheaf ω_X, and so to recover the form of Serre duality we need to shift by $n = \dim X$. Plugging all of these simplifications in, we see that we arrive at

$$H^i(X, \mathcal{F}^\vee \otimes \omega_X) \cong H^{n-i}(X, \mathcal{F}).$$

A vast generalization of this is due to Grothendieck and Verdier. At the time there was great interest in formulating a relative version of Serre duality. Indeed the absolute version looks a little like an adjunction, and if you forget the indices and treat Ext as Hom, then it *looks like* an isomorphism

$$\mathrm{Hom}(\mathbf{R}f_*\mathcal{F}, \mathcal{E}) \to \mathrm{Hom}(\mathcal{F}, f^!\mathcal{E}).$$

The trouble is of course to make this intuition meaningful. Grothendieck announced at his ICM talk that such a thing could be done, but the necessary homological algebra had not yet been developed. This was the original motivation for the derived category. There is a lot to say here, so we will just give the result and refer to [38] for a modern proof.

Theorem 4.7 (Grothendieck-Verdier duality). *Let $f : X \to Y$ be a proper morphism of schemes of finite type over a field. Then there exists a right adjoint $f^! : \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$ to the functor $\mathbf{R}f_*$ and a morphism*

$$\theta_f : \mathbf{R}f_*\mathbf{R}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{E}) \to \mathbf{R}\mathscr{H}om(\mathbf{R}f_*\mathcal{F}, \mathbf{R}f_*f^!\mathcal{E})$$

whose composition with the natural map of the adjunction:

$$\mathbf{R}\mathscr{H}om(\mathbf{R}f_*\mathcal{F}, \mathbf{R}f_*f^!\mathcal{E}) \to \mathbf{R}\mathscr{H}om(\mathbf{R}f_*\mathcal{F}, \mathcal{E})$$

is an isomorphism, and is functorial in both arguments.

The natural map in the theorem above which arises from the adjunction is sometimes called a *trace* map. Here is an example which of interest to us. Let $f : X \to Y$ be a proper, smooth morphism of relative dimension n. Then can show that $f^!(\mathcal{E}) = f^*(\mathcal{E}) \otimes \omega_{X/Y}$, where $\omega_{X/Y}$ is called the relative canonical sheaf.

4.2.2. *DG Categories.* We should pause here to remark that there are also several new approaches to the foundations which lead to a large number of improvements and strong results. While the derived category of an abelian category is certainly a remarkable object, there are a number of ways in which the derived category is problematic; one could say that in passing to this localization one forgets too much data. For example: it does not have limits or colimits (In fact one can show the existence of the weaker notion of homotopy limits or colimits, but the derived category with only the triangulated structure does not give a prescription for how to construct them.), and the existence of the kernel or cokernel of a morphism is not guaranteed. Further, the derived category as we have constructed it is unequivocally useless for local computations.

One such improvement is that of DG (differential graded) categories. This approach begins with the observation that $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B)$ is actually a complex, with degree n term $\mathrm{Hom}_{K(\mathcal{A})}(A, B[n])$. The differential on this complex is so that the zeroth cohomology is exact the set of morphisms in $K(\mathcal{A})$. In general, a DG module over a commutative ring is a $\mathbb{Z}$-graded complex of modules equipped with a differential, and the morphisms here are the obvious ones. Then a DG category is a category which is enriched over the category of such DG modules, i.e., all the morphism spaces are differential graded complexes. As we mentioned above, if C is a DG category, there is an associated category $H^0(C)$ whose objects are the same and the morphism spaces are the zeroth cohomology groups. The category $H^0(C)$ is called the homotopy category, in honor of the case where $C = R$-Mod, then $H^0(C) = K(R\text{-Mod})$. In this case, we say that C is a *DG ehancement* of $H^0(C)$.

It is not clear that we should be able to find a meaningful DG enhancement of an arbitrary triangulated category, but in specific cases, we can often find natural DG enhancements. As in the above for any ring R we can define the DG category $\mathcal{C}(R\text{-Mod})$ so that it gives an enhancement of the homotopy category $H^0(\mathcal{C}(R\text{-Mod})) = K(R\text{-Mod})$. In general, other derived categories of interest, such as derived categories of quasi-coherent or coherent sheaves on a scheme X, possess similar dg enhancements. In many useful cases as well, they are unique [33]. Indeed consider the DG category $\mathcal{C}(\mathrm{Qcoh}\, X)$ of unbounded complexes of quasi-coherent sheaves on a variety X, and the full DG subcategory $\mathcal{A}(\mathrm{Qcoh}\, X)$ spanned by all the acyclic complexes. The quotient

$\mathcal{D}(\mathrm{Qcoh}\, X) = \mathcal{C}(\mathrm{Qcoh}\, X)/\mathcal{A}(\mathrm{Qcoh}\, X)$ is a DG enhancement of the category $\mathrm{D}(\mathrm{Qcoh}\, X)$ (note that we have denoted DG categories by calligraphic letters).

All of this is a mild increase in complexity, but the payoff seems to be enormous. All of sudden, all functors between derived categories of varieties come from objects on the product (integral transforms, which we will describe later), there are "categorical resolutions of singularities", derived categories can be glued together and computations can be done locally, and they control the higher K-theory of the category. There are further pushes to work with ∞-categories, but there is no hope of summarizing this extremely active area, and so we will say nothing.

Some Final Remarks. While we have tried to make the above a fairly accessible and useful introduction to the subject of derived categories, there are a near innumerable number of topics which we have omitted. Any true devotee of the subject should read the excellent book by Huybrechts [23], followed by more specialized literature. We hope that any reader learning derived categories for the first time found our account to be readable at the very least.

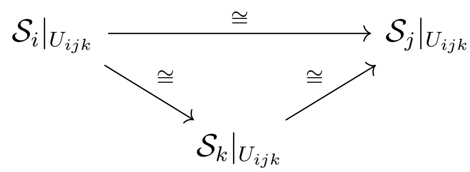
5. Equivariant Sheaves

We would now like to turn our attention to the equivariant setting. There are several references to this material. An introduction to this material (and what the first part of the following discussion follows) can be found in [1]. A more serious introduction in the topological case with constructible sheaves can be found in [2], and a good introduction to equivariant sheaves can be found in the introduction to Chapter 5 in [6], and we use all of these resources heavily. In perhaps more modern language one would use quotient stacks here, but we hav elected to stay as concrete as possible, at the expense of a bit of tedium in some instances.

Let G be an algebraic group (see Definition 2.1) and X a smooth complex variety (quasi-projective), we denote by $m : G \times G \to G$ and $e : \mathrm{Spec}\, k \to G$ the product and identity respectively, and by $s : X \to \mathrm{Spec}\, \mathbb{C}$ the structure map. Now endow X with a G-action, so that it is a G-variety in the sense of Definition 2.2. If $g \in G$ is a closed point, we also regard it as the composition $X \cong \{g\} \times X \to G \times X \stackrel{\sigma}{\to} X$.

We now want to discuss what is means for a sheaf to be equivariant. Typically when one wants to study equivariant objects on a space X, it is equivalent to studying the corresponding non-equivariant objects on X/G (or $[X/G]$). In our case however, with X a variety and G an algebraic group, such a quotient may not exist or be nice. So in order to try and capture this, we need to be more creative. Let us try to motivate equivariant sheaves in terms of descent theory [13].

Consider the following problem: given an algebraic variety X, and an open affine cover U_i, with sheaves $\mathcal{S}_i$ on each U_i, when is there a global sheaf $\mathcal{F}$ on X such that $\mathcal{F}|_{U_i} \cong S_i$? Suppose for a moment that such a sheaf existed. Then on the intersections, $\mathcal{F}|_{U_i \cap U_j} \cong \mathcal{F}|_{U_j \cap U_i}$, and so it is certainly a necessary condition that we are equipped with isomorphisms $\phi_{ij} : \mathcal{S}_i|_{U_i \cap U_j} \to \mathcal{S}_j|_{U_i \cap U_j}$. This by itself isn't enough to furnish such an $\mathcal{F}$ however, as the definition of a sheaf requires one to factor restrictions through two maps. In addition the isomorphisms above should be transitive with respect to triple intersections (so that if we were to specify an equivalence relation, we would actually get one). Hence another issue lies in the triple intersections. On $U_{ijk} = U_i \cap U_j \cap U_k$, we see that $\mathcal{S}_i|_{U_{ijk}}$ $\mathcal{S}_j|_{U_{ijk}}$, and $\mathcal{S}_k|_{U_{ijk}}$ are all pairwise isomorphic, forming the triangle:

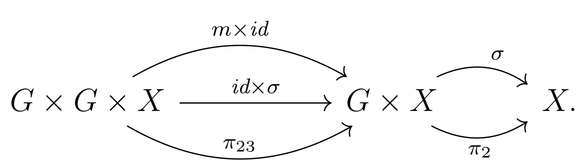


We would like this triangle to commute, i.e., $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on U_{ijk}. Then once we have these two conditions enforced, it is a striaghtforward exercise to show that $\mathcal{F}$ can be constructed (see for example, [20]).

Lets now try to rephrase these conditions in a more relative setting. Consider the variety (no longer irreducible), $X' = \coprod U_i$, and let $\pi : X' \to X$ be the map which is the inclusion on each U_i. Now it should be clear that to give a sheaf $\mathcal{S}_i$ on each U_i as above, this is the same as giving a single sheaf $\mathcal{S}$ on X'. The question can now be rephrased as: does there exist a sheaf (of say, $\mathcal{O}_X$-modules) $\mathcal{F}$ on X, such that $\pi^*\mathcal{F} \cong \mathcal{S}$. Now the fiber product $X' \times_X X' = \coprod U_i \cap U_j$, and so if p_1 and p_2 are the respective projections, the double intersection property can be rephrased as a given isomorphism $\phi : p_1^*\mathcal{F} \to p_2^*\mathcal{F}$. Now similarly to the above the triple product $X' \times_X X' \times_X X' = \coprod U_i \cap U_j \cap U_k$, and so denoting by p_{ij} the projection to the i, jth factors, the triple intersection property asks whether the given isomorphism ϕ is compatible on triple intersections: $p_{13}^*(\phi) = p_{23}^*(\phi) \circ p_{12}^*(\phi)$, where the upper asterisk indicates the pullback functor on sheaves. This condition is called the cocycle condition.

Given a quasi-coherent sheaf $\mathcal{F}$ on X', together with the isomorphism ϕ satisfying the cocycle condition is called a sheaf with descent datum, and a morphism of such sheaves $(\mathcal{F}, \phi) \to (\mathcal{G}, \psi)$ is a morphism that commutes with ϕ and ψ on $X' \times_X X'$. It is a theorem of Grothendieck that this can be generalized outside of the scope of simple glueing applications, namely if $f : X' \to X$ is a so-called fpqc morphism, then the category of quasi-coherent sheaves on X is equivalent to the category of quasi-coherent sheaves on X', together with descent datum. For an equivariant sheaf, the idea is to mimic this procedure. If we now let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules, then we have the following definition

Definition 5.1. *Let G be an algebraic group acting on a variety X via the action map $\sigma : G \times X \to X$, and consider the following diagram:*



Then an equivariant sheaf $\mathcal{F}$ is a pair consisting of a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X and an isomorphism $\psi : \sigma^*\mathcal{F} \to \pi_2^*\mathcal{F}$ subject to the cocycle condition

$$(m \times id)^*(\psi) = \pi_{23}^*(\psi) \circ (id \times \sigma)^*(\psi).$$

If $\mathcal{F}$ is coherent, then this is referred to as a G-equivariant coherent sheaf, and we call the resulting category of G-equivariant coherent sheaves $\mathrm{Coh}_G X$ *(or* $\mathrm{Qcoh}_G X$ *if we are dealing with quasi-coherent sheaves).*

The structure sheaf $\mathcal{O}_X$ and canonical sheaf ω_X (when defined) are always G-equivariant, for the simple reason that G acts by automorphisms, and both $\mathcal{O}_X$ and ω_X are intrinsic to the variety itself. The situation in general can be complex, so to greatly simplify the discussion let us now assume that the group G is finite (in practice many of the things we discuss work when the group scheme is *discrete*), made algebraic as a constant group scheme. In this case, denote by $g \times \mathrm{id}$ the composition $X \cong \{g\} \times X \to G \times X$, and note that per our notation above, $\sigma \circ (g \times \mathrm{id}) = g$ and $\pi_2 \circ (g \times \mathrm{id}) = \mathrm{id}$. Hence for a G-equivariant sheaf $\mathcal{F}$, pulling back along $(g \times \mathrm{id})$ yields (for the inverse isomorphism) $(g \times \mathrm{id})^* \psi^{-1} : \mathcal{F} \to g^* \mathcal{F}$. This is a family of isomorphisms denoted by $\psi_g : \mathcal{F} \to g^* \mathcal{F}$. This family of course needs to satisfy a few requirements. By the definition of an action by an algebraic group, the morphism ψ_e must be the identity, and since

$$\psi_{gh} : \mathcal{F} \to (gh)^* \mathcal{F} = h^*(g^*)\mathcal{F},$$

we see that $\psi_{gh} = h^*(\psi_g) \circ \psi_h$

From now on, we will understand that an equivariant sheaf is a sheaf with the extra datum of a family of morphisms satisfying the above (for example, written $(\mathcal{F}, \psi)$). This takes care of the objects, but to actually define the category completely we need to give the morphisms.

Definition 5.2. *Let $(\mathcal{F}, \psi)$ and $(\mathcal{E}, \phi)$ be two G-equivariant sheaves. Then a morphism of equivariant sheaves $f : (\mathcal{F}, \psi) \to (\mathcal{E}, \phi)$ is a morphism of sheaves such that*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{F} \\ \downarrow{\scriptstyle \psi_g} & & \downarrow{\scriptstyle \phi_g} \\ g^*\mathcal{E} & \xrightarrow{g^*(f)} & g^*\mathcal{F} \end{array}$$

commutes.

Equivalently, we can define a G-action on the morphism sets $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E})$ via $g \cdot f = \phi_g^{-1} \circ g^*(f) \circ \psi_g$. Then a morphism of G-equivariant sheaves is simply a morphism of sheaves that is *invariant* with respect to this action. Hence we see that

$$\mathrm{Hom}_G((\mathcal{F}, \psi), (\mathcal{E}, \phi)) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E})^G,$$

where we take invariants with respect to the action detailed above. This completes the definition of the category of equivariant sheaves, and we denote the category of quasi-coherent (coherent) equivariant sheaves by $\mathrm{Qcoh}_G X$ ($\mathrm{Coh}_G X$ respectively).

Proposition 5.1 (Proposition 5.1.2, [16])**.** $\mathrm{Qcoh}_G X$ *is an abelian category and has enough injectives.*

Example:

Let $\mathbb{1}$ denote the trivial group scheme. Then we have the following equivalences $\operatorname{Qcoh}_{\mathbb{1}} X \cong \operatorname{Qcoh} X$ and $\operatorname{Coh}_{\mathbb{1}} X \cong \operatorname{Coh} X$. The (almost tautological) proof of this is left to the reader.

An important and relevant example of this is where the action of a finite group G is trivial on X. Although at first glance this might seem boring, this is a surprisingly rich case, and is worth examining more fully.

Trivial Actions. If G is a finite group acting trivially on X (this means that $\sigma : G \times X \to X$ is just the projection map), then the structure of a G-equivariant sheaf $\mathcal{E}$ is just a representation $G \to \operatorname{Aut}_{\mathcal{O}_X}(\mathcal{E})$, that is, an action of G on $\mathcal{E}$. Indeed each isomorphism $\psi_g^{\mathcal{E}} : \mathcal{E} \to g^*\mathcal{E}$ is a morphism of equivariant sheaves, but we need to know what the sheaf $g^*\mathcal{E}$ is. By definition, $g^*\mathcal{E} = g^{-1}\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X = g^{-1}\mathcal{E}$. This latter sheaf is precisely $\mathcal{E}$, as it is the sheaf associated to the presheaf

$$U \mapsto \lim_{V \supset g(U)} \mathcal{E}(V) = \mathcal{E}(g(U)).$$

But since the action is trivial, this is precisely $\mathcal{E}(U)$. Hence an equivariant structure is precisely a "representation" $G \to \operatorname{Aut}_{\mathcal{O}_X}(\mathcal{E})$. Indeed every sheaf on X in thise case can be endowed with an equivariant structure, simply by taking $\psi_g : \mathcal{E} \to g^*\mathcal{E}$ to be the identity map. This makes sense by the above discussion.

We wish to pause here for a moment to warn the reader of a possibly misleading-misconception. Namely the difference between the notion of *invariant* and *equivariant*. To illustrate this, let $\mathcal{E}$ be a locally free sheaf and $\operatorname{Sym}(\mathcal{E}^\vee)$ the symmetric algebra of its dual. Consider the vector bundle given by $\mathbb{V}(\mathcal{E}) = \underline{\operatorname{Spec}}(\operatorname{Sym}(\mathcal{E}^\vee)) \to X$. The structure of a G-equivariant sheaf on $\mathcal{E}$ is the data of a family of isomorphisms $\psi_g : \mathcal{E} \to g^*\mathcal{E}$ satisfying the cocycle condition, which clearly amounts to giving a vector bundle isomorphism (by taking the transpose and the induced map on symmetric algebras) $g^*\mathbb{V}(\mathcal{E}) = X \times_X \mathbb{V}(\mathcal{E}) \to \mathbb{V}(\mathcal{E})$ (the former being the base change along $g : X \to X$). This then determines a morphism of vector bundles such that: the projection $E \to X$ commutes with the respective G-actions, that is, $g \in G$ takes E_x to $E_{\sigma(g,x)}$, and the map $E_x \to E_{\sigma(g,x)}$ for any g is a linear map of affine spaces. Conversely, taking the sheaf of sections of $\mathbb{V}(\mathcal{E})$, we obtain an equivariant sheaf, isomorphic to $\mathcal{E}$ with its equivariant structure (we leave it to the reader to prove this). The lesson here is that given an action of an algebraic group, the notion of invariance is intrinsic once the action has been fixed, while an equivariant structure is an extra set of data that needs to be added "by hand". This data need not be unique, and it is possible to have a nontrivial equivariant structure on a sheaf even when the action on the space is itself trivial.

When G is the trivial group we see that an equivariant structure is essentially nothing, but when the group is not trivial, we in general do not get an equivalence $\operatorname{Qcoh}_G X \cong \operatorname{Qcoh} X$ (or for coherent sheaves) even when the action by G on X is trivial. There are two such functors which send a G-equivariant sheaf to an ordinary sheaf, the first being the forgetful functor (which we will see later is a special case of the restriction functor). The other is invariants, which only makes sense when the G action on X is trivial. As the morphisms ψ_g preserve stalks, we can define the functor of "taking invariants" $[-]^G : \operatorname{Coh}_G X \to \operatorname{Coh} X$, which one can check is well defined. Indeed as before we have that the equivariant structure is a representation of G, but since we are taking invariants, the isomorphisms on open sets $(\psi_g^{\mathcal{E}^G})_U : \mathcal{E}^G(U) \to \mathcal{E}^G(U)$ must be trivial. Hence the global isomorphism must be the identity, this of course is the same as saying that $\mathcal{E}^G$ has an equivariant structure under the group $\mathbb{1}$, which we know is just an ordinary sheaf. Even better, so long as the group is finite, $[-]^G$ is an exact functor (this follows from an argument using the "averaging operator" $|G|^{-1}\sum g^*$, which applies in any scenario where taking invariants makes sense).

The Representation Category

Consider when $X = \operatorname{Spec}\mathbb{C}$. Then since this variety is affine, $\operatorname{Coh} X = \operatorname{Vect}_{\mathbb{C}}$, namely finite dimensional vector spaces over $\mathbb{C}$. Since this variety is just a point, every sheaf is constant, equal to its stalk at the unique point. Hence by the above discussion, the structure of a G-equivariant sheaf $\tilde{V}$ on X is the same thing as a representation of G on a vector space V, where $V = \Gamma(X, \tilde{V})$. To see that the morphisms are the same, note that by the definition of $\operatorname{Hom}_{\operatorname{Coh}_G X}((\mathcal{F}, \psi), (\mathcal{E}, \phi))$, any morphism must satisfy

$$\phi_g \circ f = f \circ \psi_g$$

as the action is trivial (hence $g^*(f) = f$). But taking global sections, this is precisely the condition that f_X be a morphism of representations. Since X is a point, this is clearly fully faithful.

All together then, we have shown that $\operatorname{Coh}_G(\operatorname{Spec}\mathbb{C}) \cong \operatorname{Rep}(G)$, the category of finite dimensional representations of G. We leave it to the reader to check the equivalent picture for representations of $\mathbb{C}[G]$ (as algebras).

We can use the above example to define a functor $\operatorname{Qcoh} X \to \operatorname{Qcoh}_G X$ (we also leave it to the reader to show the version of the example above for quasi-coherent sheaves). In the literature this is sometimes denoted as $(-) \otimes \eta$. In particular, given a group representation $\eta : G \to \operatorname{GL}(V)$, V some (possibly infinite dimensional) vector space, we can view this as an equivariant sheaf on $\operatorname{Spec}\mathbb{C}$ by the above. Given our variety X, recall we denoted by $s : X \to \operatorname{Spec}\mathbb{C}$ the structure map. So define η on X to be the equivariant sheaf $s^*\eta$, where η is a representation of G, regarded as a G-equivariant sheaf on $\operatorname{Spec}\mathbb{C}$. We won't show that this is equivariant, but the impatient reader is encouraged to go see how equivariant pullback is constructed, using the group homomorphism $\mathrm{id} : G \to G$. Given some non-equivariant sheaf $\mathcal{E}$ on X, we can endow it with an equivariant structure by applying the functor $\mathcal{E} \mapsto \mathcal{E} \otimes \eta = \mathcal{E} \otimes s^*\eta$. We should actually check that this gives an equivariant sheaf. Since the action is trivial, we set all morphisms $\psi_g : \mathcal{E} \cong g^*\mathcal{E}$ to be the identity, and so we can define a map

$$\mathcal{E} \otimes \eta \to g^*(\mathcal{E} \otimes \eta)$$

by first applying $\mathrm{id} \otimes g^*$ and then applying the canonical isomorphism $\mathcal{E} \to g^*\mathcal{E}$ tensored with the identity. Hence $\mathcal{E} \otimes \eta \in \operatorname{Qcoh}_G X$ (and similar for coherent sheaves). Further, the pair $[-]^G$ and $(-) \otimes \mathbb{C}[G]$, where $\mathbb{C}[G]$ is the regular representation of G, are adjoint functors (we write $[-]^G \dashv (-) \otimes \mathbb{C}[G]$), that is,

$$\mathscr{H}om_{\operatorname{Qcoh} X}(\mathcal{E}^G, \mathcal{F}) \cong \mathscr{H}om_{\operatorname{Qcoh}_G X}(\mathcal{E}, \mathcal{F} \otimes \mathbb{C}[G]).$$

We will generalize all of this later.

Finally, since in the case of a trivial action an equivariant structure preserves stalks, each stalk is then a $\mathbb{C}[G]$ module (as the action on $\mathcal{O}_{X,x}$ is trivial, and $\mathcal{O}_{X,x}$ is a $\mathbb{C}$-algebra). Hence this gives a version of Maschke's theorem for G-equivariant sheaves. Indeed given an equivariant sheaf $\mathcal{E}$, it has a decomposition into "simple" objects S_i, i.e., it must globally decompose

$$\mathcal{E} \cong \bigoplus \mathcal{S}_i,$$

where each "simple" sheaf $\mathcal{S}_i$ carries an action of G according to the irreducible representation which determines the decomposition. Hence we can write

$$\mathcal{E} \cong \bigoplus \mathcal{E}_i \otimes \rho_i,$$

where again ρ_i are representations of G, and $\mathcal{E}_i$ are ordinary sheaves. A special case of this is the tensor product $\mathcal{E} \otimes \mathbb{C}[G]$ of an equivariant sheaf and the sheaf determined by the regular representation. Using the decomposition above, this is easily seen to be the identity functor on $\operatorname{Qcoh}_G X$, as tensoring a $\mathbb{C}[G]$ module with $\mathbb{C}[G]$ is the identity. To make this even more transparent, on a G-variety X over $\mathbb{C}$, one can show that there is a sheaf of algebras $\mathcal{O}_X[G]$ such that the category of $\mathcal{O}_X[G]$ modules is equivalent to the category of G-equivariant $\mathcal{O}_X$ modules. We leave it as an exercise to the reader to explain why is $s : X \to \operatorname{Spec} \mathbb{C}$ is the structure map, why $\mathbb{C}[G] = s^*\rho_{reg}$ is precisely $\mathcal{O}_X[G]$.

5.1. **Back and Forth.** Unlike the ordinary categories of sheaves, we cannot simply push and pull lightly. We must be careful that wherever we end up, we have some way of endowing our target with the appropriate equivariant structure. As we will see, this is not a trivial task in general. In order to do anything however, the first notion that we need is a morphism between varieties with a group action. This is handled in the following.

Definition 5.3. *Let X be a G-variety, and Y a H-variety. Given a morphism $f : X \to Y$ of varieties and a group homomorphism $\varphi : G \to H$, we say that f is φ-equivariant (or just equivariant if no confusion can arise) if the diagram*

$$\begin{array}{ccc} G \times X & \xrightarrow{\varphi \times f} & H \times Y \\ \downarrow{\scriptstyle \sigma_X} & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

commutes.

Note that if X is a G-variety and Y a H-variety, then the product $X \times Y$ is naturally a $G \times H$-variety. The first pair of functors that we will address are those of restriction and induction.

Induction and Restriction. These are two of the functors which illustrate the essential differences between the functors between equivariant categories and ordinary categories. Given a subgroup $H < G$, we would like to define functors between the categories

$$\mathrm{Qcoh}_G X \rightleftarrows \mathrm{Qcoh}_H X.$$

This is accomplished by the *Restriction functor* $\mathrm{Res}_G^H : \mathrm{Qcoh}_G X \to \mathrm{Qcoh}_H X$ and the *Induction functor* $\mathrm{Ind}_H^G : \mathrm{Qcoh}_H X \to \mathrm{Qcoh}_G X$. Defining these functors is not difficult. Indeed by assumption since H was a subgroup of G, the restriction functor Res_G^H is simply a "partial" forgetful functor, defined by $\mathrm{Res}_G^H((\mathcal{E}, \psi_g)_{g\in G}) = (E, \psi_g)_{g\in H}$. This functor is trivially exact, and hence when we pass to the derived category, this functor need not be derived.

The induction functor is slightly more complicated. Let $H\backslash G$ denote the space of right cosets for H, and choose some transversal (a collection of representatives $\{g_\sigma\}$, one for each coset Hg). Then given an H-equivariant sheaf $(\mathcal{F}, \phi_h)$, we define

$$\mathrm{Ind}_H^G((\mathcal{F}, \phi_h)) = \bigoplus_{\sigma\in G/H} g_\sigma^* \mathcal{F}.$$

We should check that this is indeed G-equivariant. Indeed consider

$$g^*(\mathrm{Ind}_H^G(\mathcal{F})) = \bigoplus_{\sigma\in G/H} g^* g_\sigma^* \mathcal{F} = \bigoplus_{\sigma\in G/H} (g_\sigma g)^* \mathcal{F}.$$

Since we can write $g_\sigma g = h(g) g_{\tau(g)}$ for some unique $h(g) \in H$ and coset τ, we have that

$$\bigoplus_{\sigma\in G/H} (g_\sigma g)^* \mathcal{F} = \bigoplus_{\sigma\in G/H} (h(g) g_{\tau(g)})^* \mathcal{F} = \bigoplus_{\sigma\in G/H} g_{\tau(g)}^* h(g)^* \mathcal{F} \cong \mathrm{Ind}_H^G(\mathcal{F}),$$

where the last isomorphism was obtained via $\oplus\phi_{h(g)}^{-1}$. Hence we see that the equivariant structure on this sheaf is given via permutations and the given isomorphisms $\{\phi_h\}$.

We will see another construction of the induction functor later which will provide a proof that the induction functor is exact, and furthermore a sheaf-theoretic version of Frobenius reciprocity, after we have introduced the other related functors.

Pullback. To pullback an equivariant sheaf, we need an equivariant map. So then assume that $f : X \to Y$ is an φ-equivariant map between the G-variety X and H-variety Y. Then the pullback of a H-equivariant sheaf $\mathcal{F}$ can be given the structure of a G-equivariant sheaf via $f^*\mathcal{F} : f^*\mathcal{F} \to g^*(f^*\mathcal{F}) = f^*(\varphi(g)^*\mathcal{F})$, where we use the fact that f was equivariant to define the G action in terms of the H action. This then gives a functor $\mathrm{Qcoh}^H Y \to \mathrm{Qcoh}_G X$, which in particular this turns out to be a right exact functor on the category of (equivariant) quasi-coherent sheaves, and descends to a functor between the categories of equivariant coherent sheaves.

Pushforward. As before assume that $f : X \to Y$ is an φ-equivariant map between the G-variety X and H-variety. We may attempt to do something similar as the above, namely use the G action to determine the H action via $h^* f_* \mathcal{F} = f_*(g^*\mathcal{F})$, yet this assumes that there is a g mapping under φ to h. For this to work (i.e. to give the required morphisms for all $h \in H$) we need to assume that φ is surjective, otherwise we do not get the desired equivariant structure. Once we do that, we still have another problem, namely there may be many elements g which map to a given h, so our equivariant sheaf on Y will still have a G-equivariant structure, and the structure maps associated to a single $h \in H$ may vary with $g \in \varphi^{-1}(h)$. To fix this, set $K = \operatorname{Ker}\varphi$, and let $\mathcal{E}$ be a G-equivariant sheaf on X. Now as we discussed the normal pushforward $f_*\mathcal{E}$ is still G-equivariant, yet $K < G$ acts trivially on Y (by definition, as Y is a H-variety). Hence it is possible to take invariants with respect to K, and we define the φ-equivariant pushforward as $f_*^K\mathcal{E} = (f_*\mathcal{E})^K$. This then again defines a left exact functor between the categories $\operatorname{Qcoh}_G X \to \operatorname{Qcoh}^H Y$, called the equivariant direct image. If in addition f is proper, then it gives a functor on the categories of equivariant coherent sheaves.

If φ is not surjective, then we can define the *induced pushforward* f_{G*}^H, defined to be the composition $\operatorname{Ind}_{\varphi(G)}^H \circ f_*^{\operatorname{Ker}\varphi}$. Further the two functors f_*^K and f^* are still adjoint. Indeed for $\mathcal{E} \in \operatorname{Coh}_G X$ and $\mathcal{F} \in \operatorname{Coh}_H Y$, and a φ-equivariant morphism $f : X \to Y$, we construct the morphisms which give the adjunction, i.e., $\mathcal{F} \to f_*f^*\mathcal{F}$ and $f^*f_*\mathcal{E} \to \mathcal{E}$ in the same way as in the ordinary setting. Then we claim that these have image inside the subsheaf of invariants already. Hence taking invariants does not alter the adjunction.

Hom and Tensor. If we have two equivariant sheaves $\mathcal{F}$ and $\mathcal{E}$, the internal Hom $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ also inherits the structure of an equivariant sheaf. Indeed fix the morphisms $\psi_g : \mathcal{E} \to g^*\mathcal{E}$ and $\phi_g : \mathcal{F} \to g^*\mathcal{F}$. Then we have a morphism

$$\theta_g : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \to g^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}),$$

defined by the composition $\theta_g = \phi_g \circ g^*(f) \circ \psi_g^{-1}$ for some $f \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})(U)$ on every invariant open subset $U \subset X$. Taking global sections, it follows from the definitions that $\operatorname{Hom}_{\operatorname{Qcoh}_G X}(\mathcal{E},\mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})^G$. In particular, we see that $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{F}) = \Gamma(X,\mathcal{F})^G$ (regardless oof how G acts on X). We leave it to the reader to show that we get for free equivariant structures on both kernel and cokernels of maps, hence giving more evidence that we are dealing with an abelian category.

Now given two equivariant sheaves, $\mathcal{F}$ and $\mathcal{E}$, the tensor product the structure of an equivariant sheaf, by simply taking the tensor product of the isomorphisms given in the definition. That is, $\phi_g \otimes \psi_g : \mathcal{F} \otimes \mathcal{E} \to g^*(\mathcal{F} \otimes \mathcal{E})$. One can check that the normal adjunction between Hom and $\otimes$ also holds here.

Example:

A particularly interesting example is the following. Let G be a finite group acting freely on a smooth quasi-projective variety X. Then there is a geometric quotient X/G, which is also smooth and projective. The projection map is surjective and equivariant with respect to the action (as G acts trivially on X/G), and hence one can check that the two morphisms $\pi^* : \operatorname{Coh} X/G \to \operatorname{Coh}_G X$ and $\pi_*^G : \operatorname{Coh}_G X \to \operatorname{Coh} X/G$ (where we identify $\operatorname{Coh} X/G$ with $\operatorname{Coh}_{\mathbb{1}} X/G$) are equivalences of categories. To show this, we follow the proof in [35].

Now from the adjunction above between equivariant pushforward and pullback, we have natural morphisms $\mathcal{F} \to \pi_*^G\pi^*\mathcal{F}$ and $\pi^*\pi_*^G\mathcal{E} \to E$ on sheaves on X/G and X respectively. We simply aim to show that these are isomorphisms. Its now enough to assume that $X = \operatorname{Spec} A$ and $X/G = \operatorname{Spec} A^G$ are affine. Thus we can assume that $\mathcal{F} = M$ is simply a (finitely generated) module over A^G and similar for $\mathcal{E} = N$. Then $F \to \pi_*^G \pi^* \mathcal{F}$ translates into the map

$$S : M \mapsto (M \otimes_{A^G} A)^G,$$

and $\pi^* \pi_*^G \mathcal{E} \to E$ to

$$T : N^G \otimes_{A^G} A \to N.$$

Where M is an A^G module, and N is a G-equivariant A module (see Definition 2.8). In particular, A is faithfully flat[a] over A^G. Since the composition of unit and counit from an adjunction is the identity, we see that

$$M \otimes_{A^G} A \xrightarrow{S \otimes 1_A} (M \otimes_{A^G} A)^G \otimes_{A^G} A \xrightarrow{T} M \otimes_{A^G} A$$

must be the identity.

But by flatness, S is an isomorphism if and only if $S \otimes 1_A$ is, and hence its enough to show that all T are isomorphisms by the above. To finish this, we need to perform a trick. Suppose for right now that the quotient is a trivial "cover". This means that A is isomorphic as a ring to a finite sum $\oplus A^G$, and G acts by simply permuting the factors. Then a few moment's inspection reveal that $T : N^G \otimes_{A^G} A \cong N$, for any G-equivariant module N. Hence to reduce to the case where the quotient map is in a sense a covering, then we need to resort to completions to shrink our neighborhood small enough (a more sophisticated approach would have us take an étale neighborhood and argue from descent theory). So now for any $x \in X$, let $y = \pi(x)$, and $B = \mathcal{O}_{X/G,y}$. Hence it is enough then to show that

$$T \otimes 1_{\hat{B}} : (N^G \otimes_B A) \otimes_B \hat{B} \to N \otimes_B \hat{B}$$

is an isomorphism. However since the group acts freely, one can see that $A \otimes_B \hat{B} \cong \oplus_g \hat{\mathcal{O}}_{X,g(x)} \cong \oplus \hat{B}$, which by the previous remark finishes the proof. Thus S and T give the required natural transformation between the composition of push/pull and the identity functor.

[a] Recall that a morphism of rings $f : A \to B$ is called flat, if this morphism makes B a flat A module. That is, the endofunctor $(-) \otimes_A B$ is an exact functor. A flat morphism of rings $f : A \to B$ is called faithfully flat if the induced morphism $f^\# : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

Let us now give a new interpretation of the induction and restriction functor, as well as prove several of their mentioned properties. Consider a G-variety X and $H < G$ a subgroup. Suppose we have a sheaf $\mathcal{F}$ which is H-equivariant. We would like to induce a G-equivariant structure on this. Of course, we could use the definition above, but this is (for many reasons) a bit impractical and hard to prove things about. Instead we use the following setup

$$\begin{array}{ccc} G \times X & \xrightarrow{\pi} & G \times^H X \\ {\scriptstyle p_X}\downarrow & \searrow^{\sigma} & \downarrow{\scriptstyle \sigma} \\ X & & X \end{array}$$

In the above we have consider the product $G \times X$ and endowed it with a *free* $G \times H$-action, where the action is $(g,h)(g',x) = (gg'h^{-1}, hx)$. Experts sometimes refer to this as "liberating" the action (i.e. making it free). If we then take the quotient of this (which exists, but we will not show it) with respect to the action of H, we get $G \times^H X$, and this inherits the action morphism σ where it is instead operating on orbits and cosets. Hence $\sigma : G \times^H X$ makes sense.

Now for $\mathcal{F}$ (which is our H-equivariant sheaf), we see that $p_X^*\mathcal{F}$ is naturally a $G \times H$-equivariant sheaf, and since the quotient by a free action induces an equivalence of categories, there is a unique G-equivariant sheaf $\mathcal{F}'$ on $G \times^H X$ such that $\pi^*\mathcal{F}' = p_X^*\mathcal{F}$. Now since σ is a G-equivariant map (equivariant with respect to the identity morphism), we define $\sigma_*\mathcal{F}' = \mathrm{Ind}_H^G(\mathcal{F})$. Note that

$$G \times^H X = \coprod_{\sigma \in H/G} \{\sigma\} \times X,$$

and so the sheaf $\mathcal{F}'$ is an H-equivariant sheaf on $[G : H]$ number of copies of X, where G acts by permuting the factors. Thus this proves that this new definition agrees with the old. By the local description of $\sigma : G \times^H X \to X$, we see that σ_* is exact (the map is affine). Since the projection p_X is flat, pullback p_X^* is also exact; further any equivalence of abelian categories is exact, hence the composition is exact and Ind_H^G is an exact functor.

The proof that $\mathrm{Ind}_H^G \dashv \mathrm{Res}_G^H$ goes similarly to the case of representations of algebras, so we omit it. See [2] for a more explicit description.

5.2. **The Equivariant Derived Category.** Now of course, we want to pass to the derived category. By Proposition 5.1, the category $\mathrm{Qcoh}_G X$ has enough injectives and hence we can derive any left exact functor, but we still worry about finding an adapted class for the right exact functors. On a regular variety, we could always find a finite resolution by locally free sheaves, and hence could always derive such functors. Thankfully, the existence of a finite resolution by locally free G-equivariant sheaves is also guaranteed on a regular variety.

The difficulty in producing this resolution stems from finding a G-equivariant line bundle which is ample. If the group is finite (our case), and the variety projective, we can simply take $\mathcal{O}(1)$ and "average" by the group. However in the general case, this can be tough. Thankfully the heavy lifting can be done for a quasi-projective G-variety in this case, see Corollary 5.1.21 in [6] and references therein. As in the non-equivariant case, the presence of an ample line bundle allows one to realize that one a smooth quasi-projective variety, any G-equivariant coherent sheaf $\mathcal{F}$ is the quotient of a locally free G-equivariant sheaf. Taking the three term resolution:

$$\operatorname{Ker}\phi \hookrightarrow \mathcal{F}_1 \overset{\phi}{\twoheadrightarrow} \mathcal{F}.$$

Applying the same lemma to $\operatorname{Ker}\phi$, we get free resolution of G-equivariant sheaves, which must terminate due to Hilbert's Syzygy Theorem (on a smooth variety).

Once we have done this, we are free to pass to the derived category of $\mathrm{Qcoh}_G X$ with all of our (equivariant) derived functors

$$\mathbf{R}f_*^K, \mathbf{L}f^*, \mathbf{R}\mathscr{H}om, \text{ and } \overset{\mathbf{L}}{\otimes}$$

intact, where K refers to the kernel of a group homomorphism $\varphi : G \to H$ and f a φ-equivariant map from X to Y, as well as the functors $[-]^G$ (which is exact, hence taking invariants of a complex is the same as taking invariants of each term) if the group acts trivially. We also get Ind_H^G and Res_G^H when $H < G$ is a subgroup, as both functors are exact and so trivially extend to the derived category.

Definition 5.4. *Let X be a smooth G-variety. Then we define $\mathrm{D}^b_G(X) = \mathrm{D}^b(\mathrm{Coh}_G X)$.*

By definition, objects of $\mathrm{D}^b_G(X)$ are bounded complexes of coherent G-equivariant sheaves with invertible quasi-isomorphisms. There is a similar equivariant embedding of derived categories as in Proposition 4.4, which is due to [40]. We will give it here.

Proposition 5.2 ([40])**.** *Let $\mathcal{C} \subset \mathrm{D}^b(X)$ be the subcategory consisting of objects of $\mathrm{D}^b(X)$ equipped with a G-equivariant structure and G-invariant morphisms. That is, complexes of sheaves with isomorphisms $\phi_g : \mathcal{E} \to g^*\mathcal{E}$ satisfying the same conditions given in Definition 5.1. Then the canonical functor $\mathrm{D}^b_G(X) \to \mathcal{C}$ taking an object to itself is an equivalence of categories*

Proof. Send $D^b_G(X)$ to the subcategory $\mathcal{C}$ consisting of G-equivariant objects in $\mathrm{D}^b(X)$ ($\cong \mathrm{D}^b_{coh}(\mathrm{Qcoh}\, X)$), i.e. bounded complexes $\mathcal{E}$ in $\mathrm{D}^b(X)$ with isomorphisms $\lambda_g : \mathcal{E} \to g^*\mathcal{E}$ for all $g \in G$ satisfying the definitions for the object to be equivariant, and invariant morphisms. Since taking invariants is exact, we see that $\mathrm{Hom}_{\mathrm{D}^b_G(X)}((\mathcal{E}, \lambda), (\mathcal{F}, \mu)) = \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{E}, \mathcal{F})^G$. To show the functor is essentially surjective, taking $(\mathcal{E}, \lambda)$ in $\mathcal{C}$, choose a bounded injective resolution $\mathcal{E} \to I$, which since these are isomorphic in the derived category gives an equivariant structure on I, hence (I, λ) is an object of $\mathcal{C}$, which is a complex with an equivariant structure on each sheaf, and hence $\mathcal{C} \cong \mathrm{D}^b_G(X)$. ∎

Note the utility of this result, as by accident, we have shown that $\mathrm{D}^b_G(X)$ is equivalent to the subcategory $\mathrm{D}^b_{coh}(\mathrm{Qcoh}\, X)$ of quasi-coherent sheaves with bounded coherent cohomology *and* a G-equivariant structure. An alternative proof of this follows exactly the proof of 4.4. Indeed the primary lemma in the ordinary proof is that given a surjection of quasi-coherent sheaves $\mathcal{E} \to \mathcal{E}$, with $\mathcal{F}$ coherent, then there is a coherent subsheaf $\mathcal{E}' \subset \mathcal{E}$ such that the restriction $\mathcal{E}' \to \mathcal{F}$ is still surjective. In the equivariant setting, G-equivariant quasi-coherent sheaves are still quasi-coherent in the ordinary sense. Hence we can still find a coherent subsheaf and since the category is abelian this inherits the equivariant structure. Thus the same proof (almost verbatim) goes through.

5.2.1. *Equivariant Functors and Compatibilities.* In this section we will try to present many of the properties we have so far taken for granted in the derived category of sheaves. Notability the Grothendieck spectral sequence, base change, projection formula, etc were all "left to the reader" earlier, but in this setting we would like to have at least some satisfactory sketch of the proof. While we could also have left this "to the reader", there are some subtle issues in teh equivariant setting, so it is worth hacing a more systematic discussion. The first thing we wish to do is give the equivariant version of the Grothendieck spectral sequence.

Proposition 5.3. *Let X, Y, and Z be G, H, and K-varieties respectively. Given two surjective group homomorphisms $\varphi : G \to H$ and $\psi : H \to K$, as well as a φ-equivariant map $f : X \to Y$ and a ψ-equivariant map $g : Y \to Z$. Then we have an* *isomorphism of functors*

$$\mathbf{R}(g \circ f)_*^{\mathrm{Ker}(\psi \circ \varphi)} \cong \mathbf{R}g_*^{\mathrm{Ker}\,\psi} \circ \mathbf{R}f_*^{\mathrm{Ker}\,\varphi}.$$

The proof of this is surprisingly simple once we have checked the conditions in Theorem 4.5. All we need is that $\mathbf{R}f_*^{\mathrm{Ker}\,\varphi}$ maps its adapted class into the adapted class for $\mathbf{R}g_*^{\mathrm{Ker}\,\psi}$. However as before pushforward does not map injectives to injectives, so we need to instead use G-equivariant flasque sheaves. The notion of a G-equivariant flasque sheaf is the obvious one, so we will not belabor the point, rather we hope that every G-equivariant sheaf admits a resolution by G-equivariant flasque sheaves (which are obviously an adapted class for $f_*^{\mathrm{Ker}\,\varphi}$). The obvious construction is of course the Godement resolution. Since this is not discussed in many standard texts on algebraic geometry, we begin by reminding the reader of the Godement resolution in the non-equivariant setting.

Let $\mathcal{F}$ be a quasi-coherent sheaf, then we define a new sheaf $\mathfrak{G}^0(\mathcal{F})$ by the assignment

$$\mathfrak{G}^0(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_{X,x}.$$

Note that elements of $\mathfrak{G}^0(\mathcal{F})(U)$ are all functions $s : U \to \prod_{x \in U} \mathcal{F}_{X,x}$, such that $s(x) \in \mathcal{F}_{X,x}$. One can check (easily) that this is a sheaf, with the restriction maps being actual restriction of the functions s. Hence one can also easily check that this sheaf is flasque, and that the functor $\mathfrak{G}^0$ is exact (the diligent reader will work out what happens to the morphisms).

Now we construct an injection $\mathcal{F} \to \mathfrak{G}^0(\mathcal{F})$ in the following manner. Let U be open, then define $\mathcal{F}(U) \to \mathfrak{G}^0(\mathcal{F})(U)$ by sending an element $s \mapsto (s_x)_{x \in U}$. That this map is injective is obvious. To show now that we have a resolution by flasque sheaves, we proceed by induction, the base case being the above. So suppose we have a flasque resolution

$$0 \to \mathcal{F} \xrightarrow{\eta} \mathfrak{G}^0\mathcal{F} \xrightarrow{d^0} \mathfrak{G}^1\mathcal{F} \xrightarrow{d^1} \mathfrak{G}^2\mathcal{F} \longrightarrow \dots \xrightarrow{d^{k-1}} \mathfrak{G}^k\mathcal{F}.$$

Now define $\mathfrak{G}^{k+1}(\mathcal{F})$ to be $\mathfrak{G}^0(\operatorname{Coker} d^{k-1})$, and note that this sheaf is automatically flasque, and so we define $d^k : \mathfrak{G}^k\mathcal{F} \to \mathfrak{G}^{k+1}\mathcal{F}$ by the composition

$$\mathfrak{G}^k\mathcal{F} \to \operatorname{Coker} d^{k-1} \to \mathfrak{G}^{k+1}\mathcal{F} = \mathfrak{G}^0(\operatorname{Coker} d^{k-1}),$$

which gives the desired exactness as the second map is injective. This (in the classical setting) allows us to construct a flasque resolution of any sheaf.

Lemma 5.0.1. *The category* $\mathrm{Qcoh}_G X$ *has enough flasque sheaves.*

Proof. In the equivariant setting, this is more or less trivial. Define $\mathfrak{G}^0\mathcal{F}$ the same way and note that since $\mathcal{F}$ was a G-equivariant sheaf, so is $\mathfrak{G}^0\mathcal{F}$. Indeed since we had a family of isomorphisms $\psi_g : \mathcal{F} \to g^*\mathcal{F}$, we have isomorphisms of stalks $\psi_{g,x} : \mathcal{F}_x \to \mathcal{F}_{gx}$. Hence on every open set the isomorphisms

$$\prod \psi_{g,x} : \mathfrak{G}^0(\mathcal{F}) \to g^*\mathfrak{G}^0(\mathcal{F})$$

define an equivariant structure on the Godement sheaf. We leave it to the reader to show the cocycle condition. Further, the morphism $\mathcal{F} \to \mathfrak{G}^0\mathcal{F}$ is still injective (as a morphism of G-equivariant sheaves). To check that its a morphism of G-equivariant sheaves, denote this morphism by $f(\mathcal{F}, \psi) \to (\mathfrak{G}^0(\mathcal{F}), \prod \psi)$, then we require that $g \cdot f = \prod_{x \in X} \psi_g^{-1} \circ g^*(f) \circ \psi_g = f$, so let us compute the left hand side. We find

$$\prod_{x \in X} \psi_g^{-1} \circ g^*(f) \circ \psi_g : \mathcal{F} \to g^*\mathcal{F} \to g^*\mathfrak{G}^0\mathcal{F} \to \mathfrak{G}^0\mathcal{F},$$

which on stalks is the composite map

$$\mathcal{F}_x \to \mathcal{F}_{gx} \xrightarrow{\text{id}} \mathcal{F}_{gx} \to \mathcal{F}_x,$$

which is easily seen to be the identity. Hence the morphism $f(s) = \prod s_x$ is G-invariant. Then the same argument from here on shows the result for $\operatorname{Qcoh}_G X$. ∎

Now we can prove Proposition 5.3.

Proof of the Equivariant Grothendieck spectral sequence. Let X, Y, and Z be G, H, and K-varieties respectively. Suppose we have two surjective group homomorphisms $\varphi : G \to H$ and $\psi : H \to K$, and a φ-equivariant map $f : X \to Y$ and a ψ-equivariant map $g : Y \to Z$. Let the adapted class for the derived pushforward be G-equivariant flasque sheaves and let $\mathcal{F}$ be a complex. Then $\mathbf{R}g_*^{\operatorname{Ker}\psi} \circ \mathbf{R}f_*^{\operatorname{Ker}\varphi}(\mathcal{F})$ is computed by choosing a flasque resolution of $\mathcal{F}$, and applying the normal pushforward:

$$\mathbf{R}f_*^{\operatorname{Ker}\varphi}(\mathcal{F}) = f_*^{\operatorname{Ker}\varphi}(\mathfrak{G}^\bullet\mathcal{F}).$$

Since the resulting complex is $\mathbf{R}g_*^{\operatorname{Ker}\psi}$ adapted, we see that

$$\mathbf{R}g_*^{\operatorname{Ker}\psi} \circ \mathbf{R}f_*^{\operatorname{Ker}\varphi}(\mathcal{F}) = g_*^{\operatorname{Ker}\psi}(f_*^{\operatorname{Ker}\phi}(\mathfrak{G}^\bullet\mathcal{F})) = (g \circ f)_*^{\operatorname{Ker}(\psi\circ\varphi)}(\mathfrak{G}^\bullet\mathcal{F}) = \mathbf{R}(g \circ f)_*^{\operatorname{Ker}(\psi\circ\varphi)}(\mathcal{F}).$$

∎

There is of course a corresponding similar result for $\mathbf{L}(g \circ f)^* = \mathbf{L}f^* \circ \mathbf{L}g^*$, which we leave to the reader to work through. The above isomorphism of functors again works more generally, so long as we have the condition that the first preserves the adapted class for the second. As before, this implies the existence of a spectral sequence which computes the composition in terms of the often easier to understand smaller pieces. Lets take a look at an examples of this more general spectral sequence.

Equivariant Cohomology

Consider the composition of functors $[-]^G \circ \Gamma(X, -) = \Gamma(X, -)^G$. This is a left exact functor, and hence taking an injective resolution gives the right derived functors $\mathbf{R}\Gamma(X, -)^G : \mathrm{D}^b_G(X) \to \mathrm{Vect}_{\mathbb{C}}$. The cohomology spaces of a G-equivairant sheaf $\mathcal{F}$ are denoted by $H^i_G(X, \mathcal{F})$ and are called the G-equivariant cohomology of $\mathcal{F}$. If one were to re-build the theory of derived categories in terms of constructible sheaves (and many have, there are several canonical sources on the subject), then we could take $\mathcal{F} = \mathbb{C}_X$ the constant sheaf on X, giving the G-equivariant singular cohomology of X (in the analytic topology).

In the case of quasi-coherent $\mathcal{O}_X$-modules, the functor $[-]^G$ of taking invariants is in general left exact, and since the global sections of an injective sheaf is an injective module, we have that we can use the spectral sequence corresponding to

Proposition 5.3 to conclude that

$$E_2^{p,q} = R^p[R^q\Gamma(X,\mathcal{F})]^G \implies H_G^{p+q}(X,\mathcal{F}).$$

In the case that G is finite (more generally, we need the non-modularity condition, i.e. that $|G|$ does not divide $\operatorname{char} k$), then we know that $[-]^G$ is exact[a], hence all classical higher derived functors vanish. This immediately gives

$$H_G^i(X,\mathcal{F}) = H^i(X,\mathcal{F})^G.$$

A similar train of thought shows that when G is finite, the groups $\operatorname{Ext}_G^i(\mathcal{F},\mathcal{E}) \cong \operatorname{Ext}^i(\mathcal{F},\mathcal{E})^G$ (recall that by definition, $\operatorname{Hom}_{\operatorname{Qcoh}_G X}(\mathcal{F},\mathcal{E}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{E})^G)$. We can relate this further to things that we know. Namely recall that for any $\mathbb{Z}[G]$ module M, we can define the group cohomology with coefficients in M as the right derived functor to $M \to M^G$. Hence the above spectral sequences relate the group cohomology to the sheaf cohomology via:

$$H^p(G, H^q(X,\mathcal{F})) \implies H_G^{p+q}(X,\mathcal{F}), \qquad H^p(G,\operatorname{Ext}^i(\mathcal{F},\mathcal{E})) \implies \operatorname{Ext}_G^{p+q}(\mathcal{F},\mathcal{E}).$$

Further, if G now acts trivially, the functor $[-]^G$ is defined on sheaves, and a similar argument proves that $H_G^i(X,\mathcal{F}) = H^i(X,\mathcal{F}^G)$.

[a] This is not true for arbitrary sheaves of abelian groups.

Now let us turn to other compatibilities between derived functors. The first is is a simple generalization of the adjunction $\mathbf{L}f^* \dashv \mathbf{R}f_*$ in the ordinary case. As we will see (and this is one of the really nice things about the derived category) that many statements about nontrivial relations between sheaves (or just aren't true in $\operatorname{Coh}_G X$) turn into near trivialities due to the ability work with derived functors.

Proposition 5.4. *Let $f : X \to Y$ be a proper φ-equivariant morphism of smooth G, H-varieties respectively with $\varphi : G \to H$ a surjection. Then $\mathbf{L}f^* \dashv \mathbf{R}f_*^{\operatorname{Ker}\varphi}$, i.e.*

$$\operatorname{Hom}_{\mathrm{D}_G^b(X)}(\mathbf{L}f^*\mathcal{F},\mathcal{E}) \cong \operatorname{Hom}_{\mathrm{D}_H^b(Y)}(\mathcal{F},\mathbf{R}f_*^{\operatorname{Ker}\varphi}\mathcal{E}).$$

Proof. We may freely assume that $\mathcal{F}$ is a complex of locally free sheaves and $\mathcal{E}$ is a complex of injectives, quasi-isomorphic to $\mathcal{F}$ and $\mathcal{E}$ respectively. Then the statement reduces to the statement $f^* \dashv f_*^{\operatorname{Ker}\varphi}$, which was discussed earlier. ∎

We also have the equivariant projection formula. Which as before goes in much the same way as the ordinary case.

Proposition 5.5. *Let $f : X \to Y$ be as above. Then in $\mathrm{D}_H^b(Y)$, we have an isomorphism*

$$\mathbf{R}f_*^{\operatorname{Ker}\varphi}\mathcal{E} \overset{L}{\otimes} \mathcal{F} \cong \mathbf{R}f_*^{\operatorname{Ker}\varphi}(\mathcal{E} \overset{L}{\otimes} \mathbf{L}f^*\mathcal{F})$$

for $\mathcal{E} \in \mathrm{D}_G^b(X)$ and $\mathcal{F} \in \mathrm{D}_H^b(Y)$.

Proof. As above we can assume that $\mathcal{F}$ can be replaced by a complex of locally free sheaves and $\mathcal{E}$ replaced by injectives. Then this reduces to the statement that

$$f_*^{\operatorname{Ker}\varphi}\mathcal{E} \otimes \mathcal{F} \cong f_*^{\operatorname{Ker}\varphi}(\mathcal{E} \otimes f^*\mathcal{F}),$$

which follows from the (underived) projection formula. Technically speaking there is something to check here as it may not be clear that the projection formula holds in the equivariant setting. However the proof relies on the push-pull adjunction and a local argument, so a little care and Proposition 5.4 gives the claim. We leave the verification to the reader. ■

We hope the reader is beginning to see a pattern in how these compatibilities are proven. The next is a version of flat base change for the equivariant derived category, whose proof requires a bit more care, but is otherwise similar.

Proposition 5.6 (Equivariant cohomological base change)**.** *Consider a fiber square of (finite) algebraic groups, and a fiber square of algebraic varieties,*

$$\begin{array}{ccc} K\times_H G & \xrightarrow{\bar\psi} & G \\ \big\downarrow{\scriptstyle\bar\varphi} & & \big\downarrow{\scriptstyle\varphi} \\ K & \xrightarrow{\psi} & H \end{array} \qquad\qquad \begin{array}{ccc} Z\times_Y X & \xrightarrow{\bar g} & X \\ \big\downarrow{\scriptstyle\bar f} & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

where X is a G-variety, Y a H-variety, and Z a K-variety, and assume that φ and ψ are surjective and all morphisms of varieties are equivariant with respect to the corresponding group homomorphism. Then $\bar\varphi$ and $\bar\psi$ are surjective, with $\operatorname{Ker}\bar\varphi \cong \operatorname{Ker}\varphi$*(=$K$), and there is a morphism of functors*

$$\mathbf{L}g^*\mathbf{R}f_*^K\mathcal{E} \to \mathbf{R}\bar f_*^K\mathbf{L}\bar g^*\mathcal{E}.$$

Further, if g is flat, this is an isomorphism for any $\mathcal{E}\in \mathrm{D}^b_G(X)$.

Proof. The statement involving surjectivity and kernels of the group homomorphisms is easy to verify, so we leave it to the reader. Now the morphism of functors

$$\mathbf{L}g^*\mathbf{R}f_*^K\mathcal{E} \to \mathbf{R}\bar f_*^K\mathbf{L}\bar g^*\mathcal{E}$$

can be obtained in the following manner. By Proposition 5.4, there is a natural morphism $\mathcal{E}\to \mathbf{R}\bar g_*^{\operatorname{Ker}\psi}\mathbf{L}\bar g^*\mathcal{E}$, which induces a morphism

$$\mathbf{R}f_*^K\mathcal{E} \to \mathbf{R}f_*^K\mathbf{R}\bar g_*^{\operatorname{Ker}\bar\psi}\mathbf{L}\bar g^*\mathcal{E} \cong \mathbf{R}g_*^{\operatorname{Ker}\psi}\mathbf{R}\bar f_*^K\mathbf{L}\bar g^*\mathcal{E}.$$

Now again by Proposition 5.4, this corresponds to a morphism

$$\mathbf{L}g^*\mathbf{R}f_*^K\mathcal{E} \to \mathbf{R}\bar f_*^K\mathbf{L}\bar g^*\mathcal{E}.$$

Now we check that it is an isomorphism. Since $\mathcal{E}$ has coherent cohomology, we can check this locally, and so we can assume that g and $\bar g$ are affine morphisms (so that the pushforward is exact). Further by assumption they are flat. Then it follows that we can simply check the further induced morphism

$$g_*^{\operatorname{Ker}\psi}(\mathbf{L}g^*\mathbf{R}f_*^K\mathcal{E}) \to g_*^{\operatorname{Ker}\psi}(\mathbf{R}\bar f_*^K\bar g^*\mathcal{E}) \cong \mathbf{R}f_*^K(\bar g_*^{\operatorname{Ker}\bar\psi}\bar g^*\mathcal{E}),$$

but since tensoring with the structure sheaf is the identity functor, the projection formula tells us that checking that this is an isomorphism is the same as checking that

$$\mathbf{R}f_*^K\mathcal{E}\otimes g_*^{\operatorname{Ker}\psi}\mathcal{O}_Z \to \mathbf{R}f_*^K(\mathcal{E}\otimes \bar g_*^{\operatorname{Ker}\bar\psi}\mathcal{O}_{Z\times_Y X})$$

is an isomorphism. But this happens exactly when $\bar g_*^{\operatorname{Ker}\bar\psi}\mathcal{O}_{Z\times_Y X}\cong f^*(g_*^{\operatorname{Ker}\psi}\mathcal{O}_Z)$, which can be checked directly. ■

We list a handful of other compatibilities, which hold in both the ordinary and equivariant derived categories. We will leave it to the reader to sketch the proofs.

Proposition 5.7. *Let $f : X \to Y$ be a φ-equivariant morphism of smooth G, H-varieties respectively. Let $\mathcal{K} \in \mathrm{D}^b_G(X)$ and $\mathcal{M}$, $\mathcal{N} \in \mathrm{D}^b_H(Y)$, then*

(1) $\mathbf{L}f^*\mathcal{M} \overset{L}{\otimes} \mathbf{L}f^*\mathcal{N} \cong \mathbf{L}f^*(\mathcal{M} \overset{L}{\otimes} \mathcal{N})$,
(2) $\mathbf{L}f^*\mathbf{R}\mathscr{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}) \cong \mathbf{R}\mathscr{H}om(\mathbf{L}f^*\mathcal{M}, \mathbf{L}f^*\mathcal{N})$,
(3) *the derived dual* $\mathcal{K}^\vee = \mathbf{R}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{K}, \mathcal{O}_X) \in \mathrm{D}^b_G(X)$ *and*

$$\mathcal{K}^{\vee\vee} \cong \mathcal{K}, \qquad \mathcal{K}^\vee \overset{L}{\otimes} \mathcal{N} \cong \mathbf{R}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{K}, \mathcal{N}),$$

(4) *moreover* $\mathbf{R}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \overset{L}{\otimes} \mathcal{K} = \mathbf{R}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{K}, \mathcal{N} \overset{L}{\otimes} \mathcal{K})$.

As one can see, roughly speaking whenever there is a relation between the functors f_*^K, f^*, $\otimes$, and $\mathscr{H}om$ which holds for some special class of sheaves (locally free, flasque, etc), then so long as these sheaves form an adapted class, the resulting formula is true in the derived category for arbitrary complexes (and derived functors). We omit the remaining formal compatibilities on the grounds that they are of lesser importance for our original goal, the derived McKay correspondence. Many introductory sources on derived categories of sheaves contain a more complete list, many with complete proofs. There is one more relationship that we should dicuss, which as before is not a simple formal consequence of the definition of derived functors, namely it is the equivariant version of Grothedieck-Verdier duality, proven in this case by [38].

Theorem 5.1 (Equivariant Grothedieck-Verdier duality)**.** *Let $f : X \to Y$ be a proper φ-equivariant morphism of schemes of finite type over a field. Then there exists a right adjoint $f^{!,\mathrm{Ker}\,\varphi} : \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$ to the functor $\mathbf{R}f_*$ and a morphism*

$$\theta_f : \mathbf{R}f_*\mathbf{R}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, f^{!,\mathrm{Ker}\,\varphi}\mathcal{E}) \to \mathbf{R}\mathscr{H}om(\mathbf{R}f_*\mathcal{F}, \mathbf{R}f_*f^{!,\mathrm{Ker}\,\varphi}\mathcal{E})$$

whose composition with the natural map of the adjunction:

$$\mathbf{R}\mathscr{H}om(\mathbf{R}f_*\mathcal{F}, \mathbf{R}f_*f^{!,\mathrm{Ker}\,\varphi}\mathcal{E}) \to \mathbf{R}\mathscr{H}om(\mathbf{R}f_*\mathcal{F}, \mathcal{E})$$

is an isomorphism, and is functorial in both arguments.

As before $f^{!,\mathrm{Ker}\,\varphi}\mathcal{O}_Y$ will be the G-equivariant dualizing complex for f, and whenever f is a proper smooth map[9] of relative dimension r, $f^{!,\mathrm{Ker}\,\varphi}\mathcal{O}_Y \cong \omega_{X/Y}[-r]$, where we give $\omega_{X/Y}$ the natural G-equivariant structure. In particular, $f^{!,\mathrm{Ker}\,\varphi}\mathcal{E} = f^*\mathcal{E} \overset{L}{\otimes} \omega_{X/Y}[-r]$. If we take $f : X \to \operatorname{Spec}\mathbb{C}$ to be the structure morphism, $\mathcal{E} = \mathcal{O}_{\operatorname{Spec}\mathbb{C}}$, and $\mathcal{F}$ a locally

[9]Recall that a morphism $f : X \to Y$ of schemes of finite type over a field is said to be smooth of relative dimension r if

(1) f is flat,
(2) f maps irreducible components of X of dimension n_i into irreducible components of Y of dimension $n_i + r$, and
(3) the $k(x)$-dimension of the vector fiber $\Omega_{X/Y,x} \otimes k(x)$ of $\Omega_{X/Y}$ is r for all points $x \in X$.

free sheaf, then the isomorphism in the statement of Grothendieck-Verdier duality simplifies (via a spectral sequence) to

$$H^q_G(X, \mathcal{F}^\vee \otimes \omega_X) \cong \mathrm{Ext}^q_G(\mathcal{F}, \omega_X) \cong H^{r-q}_G(X, \mathcal{F})^*,$$

which is the classical statement of Serre duality (combined with taking G-invariants).

5.3. **The Affine Plane.** So far we have spent a considerable amount of effort to describe the basics of equivariant derived categories on a general variety. For our purposes however, we are interested purely in the case when $X = \mathbb{C}^2$, so let us turn to that now. Let G be a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, and let G act tautologically on $\mathbb{C}^2 = \mathrm{Spec}\,\mathbb{C}[x, y]$. Since $\mathrm{Qcoh}\,\mathbb{C}^2 \cong \mathbb{C}[x, y]$-Mod, we expect that $\mathrm{Qcoh}_G\,\mathbb{C}^2$ will also be a module category over some ring. Of course the right guess is that of a G-equivariant module (see Definition 2.8).

To see why this is a reasonable guess, let $\mathcal{M}$ be a G-equivariant sheaf on $\mathbb{C}^2$. Then for any $g \in G$, we have an isomorphism on global sections $\tilde{\psi}_g : \Gamma(\mathbb{C}^2, \mathcal{M} \cong \Gamma(\mathbb{C}^2, g^*\mathcal{M})$. Since the latter module is canonically the same as the former (denoting both by M), we claim that M admits a structure of a G-module. Indeed the morphism $\Psi : G \times M \to M$ defined by $\Psi(g, m) = \tilde{\psi}_g(m)$ satisfies the axioms for an action, as can be checked directly. However there is more to show; recall a G-equivariant R-module M is an R-module that has an action by G, such that the multiplication map $R \otimes M \to M$ is G-equivariant: $g(rm) = g(r)g(m)$ for all $r \in R$, $m \in M$, and $g \in G$. So now let $f \in \mathbb{C}[x, y]$ and consider $\Psi(g, fm)$. This is by definition just $\tilde{\psi}_g(fm) \in \Gamma(\mathbb{C}^2, g^*\mathcal{M}) = \mathbb{C}[x, y] \otimes_{\mathbb{C}[x,y]} M$, where of course g acts on $\mathbb{C}[x, y]$ via its tautological representation on functions on $\mathbb{C}^2$, and since everything in sight is equivariant, we see that M is an equivariant module. Comparing the two definitions of morphisms (that of equivariant sheaves and equivariant modules), we see further that this gives an equivalence of categories.

The obvious corollary to this is via Proposition 2.4,

Corollary 5.1.1. *There is an equivalence of categories*

$$\mathrm{Qcoh}_G\,\mathbb{C}^2 \cong \mathbb{C}[x, y]\text{-}Mod_G \cong \mathbb{C}[x, y]\#[G]\text{-}Mod.$$

Hence

$$\mathrm{D}^b_G(\mathbb{C}^2) \cong \mathrm{D}^b(\mathbb{C}[x, y]\#[G]\text{-}mod),$$

where the latter is the derived category of finitely generated[10] *modules over the crossed product algebra.*

This is certainly a nontrivial simplification, and can assist in giving very explicit descriptions of the derived category in many cases. More generally, we saw earlier that on a (quasi-projective) variety where the group acts trivially the equivariant $\mathcal{O}_X$ modules were simply $\mathcal{O}_X[G]$ modules, but now we see the generalization to when the group does not act trivially. Indeed if X is any variety upon which a finite group acts, then there is a coherent sheaf of algebras $\mathcal{O}_X\#[G]$ on X such that the category of $\mathcal{O}_X\#[G]$-modules is equivalent to the equivariant $\mathcal{O}_X$ modules [5]. We won't use this fact much, but it is nice to have such a clear characterization of the objects we

[10] Indicated by the lowercase "m" in mod.

are dealing with and have such a nice parallel to the representation theory of algebras which we discussed earlier.

Lets try to use this equivalence to explore more about the equivariant derived category on $\mathbb{C}^2$. The category $\mathbb{C}[x,y]\text{-Mod}_G$ has many nice properties as a benefit of being a category of modules. In particular we have that this category possesses both enough projectives and injectives. Considering the case when $G \cong \mathbb{1}$, the finitely generated projective objects in this category are precisely the free modules by the Quillen-Suslin theorem. For this reason one would expect that the presence of a nontrivial group action would only restrict this class further, i.e., there are no non-free projective objects when G is nontrivial. Indeed since G is a finite group, invariants is an exact functor, so $\text{Hom}_G(-,-) = [-]^G \circ \text{Hom}(-,-)$; moreover an object P is projective if and only $\text{Hom}(P,-)$ is exact. This immediately implies that a G-equivariant R module is projective if and only if it is projective as a R module. Another more geometric way to see this is to recall that a finitely generated projective module corresponds to an algebraic vector bundle over an affine scheme. Since these are all trivial (by the Quillen-Suslin theorem again), and an equivariant structure is *additional data*, we see that any equivariant projective module is free. Further, an elementary computation will show that any free module is projective (although the analogue for locally free sheaves and projective sheaves is strikingly false).

However we have not used the fact that a G-equivariant $\mathbb{C}[x,y]$ module is equivalently a $\mathbb{C}[x,y]\#[G]$ module, and this viewpoint gives us slightly more information, as they are modules over a larger ring. If R is any ring (say over $\mathbb{C}$) with a G action, a left $R\#[G]$ module M is nothing but an R module with a compatible action of G, in the sense of an equivariant module. Since we can define an action $(r \otimes g)(s) = rg(s)$ for all r and s in R, R itself is a left $R\#[G]$ module; of course, $R\#[G]$ is also a left module over itself. By definition, a morphism of $R\#[G]$ modules is a morphism which is $R\#[G]$ linear, so in particular we have

$$\text{Hom}_{R\#[G]}(M,N) = \text{Hom}_R(M,N)^G.$$

As before since taking invariants is exact, the spectral sequence for the composition of derived functors collapses immediately, and we get

$$\text{Ext}^i_{R\#[G]}(M,N) = \text{Ext}^i_R(M,N)^G,$$

just as with the equivariant case. This immediately proves the following:

Proposition 5.8. *M is projective as a $\mathbb{C}[x,y]\#[G]$ module if and only if M is projective as a $\mathbb{C}[x,y]$ module.*

More can be said however, we can relate $\mathbb{C}[x,y]\#[G]$ modules to representations of G. Let M be in $\mathbb{C}[x,y]\#[G]$-Mod, and let V be in $\mathbb{C}[G]$-Mod. There is a natural action of the skew group algebra on $M \otimes_{\mathbb{C}} V$ by the diagonal action: $(r \otimes g)(m \otimes v) = (r \otimes g)(m) \otimes g(v)$, which arises naturally from the group algebra $\mathbb{C}[G] \cong \prod \text{End}(V_i)$. Now define a functor $\mathbb{C}[G]\text{-Mod} \to \mathbb{C}[x,y]\#[G]\text{-Mod}$ which sends a representation V to the module $\mathbb{C}[x,y] \otimes_{\mathbb{C}} V$. For any representation V, restriction of scalars gives that $\mathbb{C}[x,y] \otimes_{\mathbb{C}} V$ is a free $\mathbb{C}[x,y]$ module, and hence projective. This induces an equivalence between the projectives in $\mathbb{C}[x,y]\#[G]$-Mod and $\mathbb{C}[G]$-Mod (in particular, since the ring is semisimple, all $\mathbb{C}[G]$ modules are projective). Further, this induces a bijection between the isomorphism classes of indecomposable projective $\mathbb{C}[x,y]\#[G]$ modules and isomorphism classes of simple representations of G.

In particular since the variety $\mathbb{C}^2$ is smooth, every quasi-coherent sheaf admits a finite resolution by locally free sheaves (= projectives in this case), the length bounded by the dimension (= 2). There is an obvious candidate for resolving such sheaves associated to a local complete intersection, namely the Koszul resolution. We had used this resolution before in an example of a spectral sequence, but let us now pause to describe it in more detail. Let $\mathcal{E}$ be a locally free sheaf of rank n on a (smooth) variety X, and let s be a global section of $\mathcal{E}$. As such s determines a morphism $\mathcal{O}_X \to \mathcal{E}$, and as such a dual morphism $d_1 : \mathcal{E}^\vee \to \mathcal{O}_X$. The image of d_1 is an ideal sheaf, which determines a closed subscheme called the zero scheme of s. We can define the Koszul complex relative to s to be the complex $K(s)$:

$$0 \to \bigwedge^n \mathcal{E}^\vee \to \bigwedge^{n-1} \mathcal{E}^\vee \to \cdots \to \bigwedge^1 \mathcal{E}^\vee \xrightarrow{d_1} \mathcal{O}_X \to 0,$$

where the morphism $d_p : \bigwedge^p \mathcal{E}^\vee \to \bigwedge^{p-1} \mathcal{E}^\vee$ is defined by

$$d_p(t_1 \wedge \cdots t_p) = \sum_{j=1}^{p} (-1)^{j-1} t_j(s)\, t_1 \wedge \cdots \wedge \hat{t_j} \wedge \cdots \wedge t_p,$$

and the hat means omission.

Take some point $x \in X$. Then the stalk $\mathcal{E}_x$ is a free module over $\mathcal{O}_{X,x}$. Taking a basis, we can represent s as a sequence $a_1, ..., a_n$ of elements of $\mathcal{O}_{X,x}$, and then the stalk of $K(s)$ is precisely the classical Koszul complex associated to a regular sequence $K(a)$ (see [12]). So we see that is x does not belong to the closed subscheme determined by s (the zero scheme of s), the sequence above is exact. On the other hand, if the sequence $(a_1, ..., a_n)$ is a regular sequence at x, then the sequence above is a finite free resolution of $\mathcal{O}_{Z(s)}$, where $Z(s)$ is the zero scheme of s. Hence, the following exact sequence is called the Koszul resolution of $\mathcal{O}_{Z(s)}$ determined by s:

$$0 \to \bigwedge^n \mathcal{E}^\vee \to \bigwedge^{n-1} \mathcal{E}^\vee \to \cdots \to \bigwedge^1 \mathcal{E}^\vee \xrightarrow{d_1} \mathcal{O}_X \to \mathcal{O}_{Z(s)} \to 0.$$

Example:

Consider $X = \mathbb{P}^n$, and let $Y \subset X$ be a hypersurface of degree d. Then by definition, Y is the zero locus of a global section of $\mathcal{O}_{\mathbb{P}}(d)$, and hence determines a morphism $\mathcal{O}_{\mathbb{P}}(-d) \to \mathcal{O}_{\mathbb{P}}$ (which is injective). By the fact that this is codimension one, we have that at any point in Y we get a regular sequence, and the Koszul resolution applied to the map gives a finite free resolution:

$$0 \to \mathcal{O}_{\mathbb{P}}(-d) \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_Y \to 0,$$

which is exactly the "standard" exact sequence for a divisor.

6. Fourier-Mukai Transforms

The first instance of a Fourier-Mukai functor was introduced by Mukai [?] when he applied it study the derived category of an abelian variety and its dual. Abelian varieties are algebraic groups (Definition 2.1) which are complete, and this additional requirement actually forces them to be abelian algebraic groups, hence the name. Over $\mathbb{C}$ with the complex topology, abelian varieties are simply torii, i.e. quotients of $\mathbb{C}^g$ by a lattice Λ. The dual of an abelian variety X is another abelian variety $\hat{X}$ which acts as a fine moduli space for degree zero line bundles on X, see the lectures by Kleiman [13] for a modern treatment. The condition to be a fine moduli space is similar to the representability of the Hilbert functor we discussed earlier; in particular it implies the existence of a *universal bundle* $\mathcal{P}$ on the product $X \times \hat{X}$ (sometimes called the Poincaré bundle) whose restriction $\mathcal{P}|_{X\times\{\xi\}}$ to $\xi \in \hat{X}$ is precisely the line bundle on X defined by ξ.

Mukai then wanted to show that $\mathrm{D}^b(X) \cong \mathrm{D}^b(\hat{X})$, the key idea is to mimic the notion of an integral transform in analysis. An integral functor between two function spaces is roughly defined to be something of the form

$$K(f)(x) = \int K(x, y) f(y)\, dy.$$

In our context, the pullback corresponds to regarding $f(y)$ as a function of both x and y, tensor product is the multiplication by $K(x, y)$, and the integral is the pushforward. In the context of abelian varieties over $\mathbb{C}$ the natural integral transform which arises is the Fourier transform, hence the name. Now if $X \xleftarrow{p} X \times \hat{X} \xrightarrow{q} \hat{X}$ are the projections, Mukai defined the functor

$$\Phi_{\mathcal{P}}^{X\to\hat{X}}(\mathcal{F}) = \mathbf{R}q_*(\mathbf{L}p^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{P})$$

which maps $\mathrm{D}^b(X)$ to $\mathrm{D}^b(\hat{X})$. Further, Mukai proved that this gave an equivalence of categories. As it eventually turned out, the particular case that Mukai was interested is in some sense rare, the possible pairs of non-isomorphic varieties with equivalent derived categories are abelian varieties, Calabi-Yau varieties, and few others.

More generally consider two varieties X and Y, and denote by $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ the projections. Now fix some object $\mathcal{E}$ in $\mathrm{D}^b(X \times Y)$, called the "kernel" of the transform. We can then define the integral functor (or integral transform) with kernel $\mathcal{E}$ as the functor

$$\Phi_{\mathcal{E}}^{X\to Y}(\mathcal{F}) = \mathbf{R}q_*(\mathbf{L}p^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{E}).$$

We can actually slightly simplify this, as since the projections are flat morphisms, p^* is exact, hence $\mathbf{L}p^* = p^*$.

Definition 6.1. *An integral transform $\Phi_{\mathcal{E}}^{X\to Y}$ is called a Fourier-Mukai transform if it is an equivalence of categories.*

Lets first tackle an example of an integral functor. As we will see, most functor that we have considered so far are hidden examples of an integral transform, and its actually surprisingly hard to find a functor of geometric interest that is not an example of one.

Here we see that a suitable choice of kernel $\mathcal{E}$ defines an integral functor which is the same as the derived pushforward $\mathbf{R}f_*$.

Example:

Consider a proper morphism $f : X \to Y$. Then the derived functor Rf_* can be expressed as the integral functor with kernel $\mathcal{O}_{\Gamma_f}$. Indeed the graph can be realized as the pushforward $(\mathrm{id} \times f)_* \mathcal{O}_X$, in the diagram below.

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{id}\times f} & X \times Y & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ X & & & & Y \end{array}$$

Note that $p_1 \circ (\mathrm{id} \times f) = \mathrm{id}$ and $p_2 \circ (\mathrm{id} \times f) = f$. Then we see that

$$\Phi^{X \to Y}_{\mathcal{O}_{\Gamma_f}}(\mathcal{F}) = \mathbf{R}p_{2*}(p_1^* \mathcal{F} \overset{\mathbf{L}}{\otimes} \mathbf{R}(\mathrm{id} \times f)_* \mathcal{O}_X).$$

Then by the projection formula, this is equal to

$$\mathbf{R}p_{2*}(\mathbf{R}(\mathrm{id} \times f)_*[\mathbf{L}(\mathrm{id} \times f)^* p_1^* \mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{O}_X]) = \mathbf{R}(p_2 \circ (\mathrm{id} \times f))_*([\mathbf{L}(\mathrm{id} \times f)^* p_1^* \mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{O}_X])$$

Which is of course $\mathbf{R}f_*(p_1 \circ (\mathrm{id} \times f)^* \mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{O}_X) = \mathbf{R}f_*(\mathcal{F})$.

As a special case of this, we see that $\Phi^{X \to X}_{\mathcal{E}}$ is isomorphic (as a functor) to the identity functor on $\mathrm{D}^b(X)$ if and only if $\mathcal{E}$ is quasi-isomorphic to $\mathcal{O}_\Delta$.

Example:

Let $Y = X$, and let $\mathcal{L}$ be a line bundle on X. Setting $\mathcal{E} = q^* \mathcal{L}$, we can check (similar to the above) that $\Phi^{X \to X}_{\mathcal{E}}(-) \cong (-) \otimes \mathcal{L}$. Note that the tensor product is underived as $\mathcal{L}$ is locally free.

It is also true that these functors are closed under composition, indeed let $\Phi^{X \to Y}_{\mathcal{E}}$ and $\Phi^{Y \to Z}_{\mathcal{K}}$ be two integral transforms. If we define the convolution of two kernels to be

$$\mathcal{E} * \mathcal{K} = \mathbf{R}\pi_{XZ*}(\pi^*_{XY}(\mathcal{E}) \overset{\mathbf{L}}{\otimes} \pi^*_{YZ}(\mathcal{K})),$$

then we claim that $\Phi^{Y \to Z}_{\mathcal{K}} \circ \Phi^{X \to Y}_{\mathcal{E}} = \Phi^{X \to Z}_{\mathcal{E} * \mathcal{K}}$. We will prove this below in the equivariant case, and the result in this context can be recovered by taking all groups to be trivial. With all of this, it is a reasonable question to ask what kinds of functors cannot be realized as integral transforms. The answer is surprising, and the following theorem is due to Orlov [39].

Theorem 6.1. *Let X and Y be two smooth projective varieties and $F : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ be a fully faithful functor of triangulated categories which admits a left and right adjoint. Then there exists a unique (up to isomorphism) object $\mathcal{E}$ of $\mathrm{D}^b(X \times Y)$ such that F is isomorphic to the integral transform $\Phi^{X \to Y}_{\mathcal{E}}$.*

More surprising, there are several generalizations of this, the strongest statements due to Töen, Lunts and Orlov (see [33] and references therein).

Another very nice fact is that in the proper setting adjoints of integral transforms are also integral transforms, and have a very explicit description.

Proposition 6.1. *Let X and Y be smooth projective varieties and $n = \dim X$, $m = \dim Y$. Then given an integral transform $\Phi_{\mathcal{E}}^{Y\to X}$, the functors $\Phi^{X\to Y}_{\mathcal{E}^\vee \otimes q^*\omega_{Y/k}[m]}$ and $\Phi^{X\to Y}_{\mathcal{E}^\vee \otimes q^*\omega_{X/k}[n]}$ are the right and left adjoints respectively.*

We encourage the reader to work through the proof of this (you will need Grothendieck-Verdier duality, which holds so long as the varieties are proper over a field). The final result which we will reference is the reconstruction result of Bondal-Orlov (a proof in [23]).

Theorem 6.2. *Let X and Y be two smooth projective varieties. If ω_X is (anti) ample and $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$, then so is ω_Y, and moreover $X \cong Y$.*

There are several proof of this result, the original is given in [23], which explicitly constructs the morphism by looking at so-called "point objects" and "invertible objects" in the derived category. The former are used to construct a homeomorphism on the underlying spaces, and the latter to construct an isomorphism between canonical rings, and taking the projective spectrum of both sides completes the proof.

Of course, if we instead work in equivariant derived categories, we need to be slightly more careful. Since (to the best of my knowledge) we do not have an equivariant version of Orlov's characterization, we say that such an equivariant integral functor which is an equivalence of triangulated categories is an *equivariant Fourier-Mukai functor* if there exists an object in $\mathcal{Q} \in \mathrm{D}^b_{H\times G}(Y \times X)$ such that the quasi-inverse functor is isomorphic to $\Phi^{Y\to X}_{\mathcal{Q}}$. If X is a G-variety and Y a H-variety, then the product $X \times Y$ is naturally a $G \times H$-variety. So now again given the projections $X \stackrel{p}{\leftarrow} X \times Y \stackrel{q}{\rightarrow} Y$ and the projections of the group $G \leftarrow G \times H \rightarrow H$. Then given some $\mathcal{P} \in \mathrm{D}^b_{G\times H}(X \times Y)$, we can similarly define $\Phi^{X\to Y}_{\mathcal{P}}(\mathcal{F})$ using the same composition of the equivariant functors

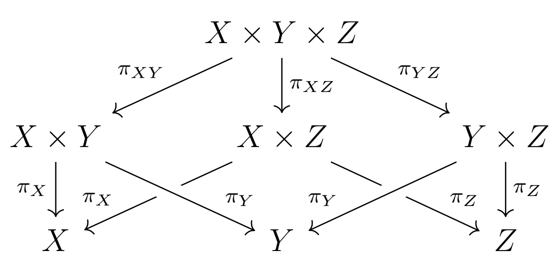
$$\mathbf{R}q_*^{G\times \mathbb{1}}(\mathbf{L}p^*(\mathcal{F}) \overset{\mathbf{L}}{\otimes} \mathcal{P}).$$

As in the non-equivariant case, such functors are closed under composition, with the composition having a kernel of the same overall form, but with the inherited equivariant structure.

Proposition 6.2. *Equivariant integral transforms (and hence equivariant Fourier-Mukai transforms) are closed under composition. Namely $\Phi^{Y\to Z}_{\mathcal{K}} \circ \Phi^{X\to Y}_{\mathcal{E}} = \Phi^{X\to Z}_{\mathcal{E}*\mathcal{K}}$, where*

$$\mathbf{R}\pi^{\mathbb{1}\times H\times \mathbb{1}}_{XZ*}(\pi^*_{XY}\mathcal{E} \overset{\mathbf{L}}{\otimes} \pi^*_{YZ}\mathcal{K}).$$

Proof. Indeed if we have G, H, K-varieties X, Y, and Z respectively, then consider the diagram:



In the above diagram, the reader should be careful to note that the maps are all labeled similarly, but no two are actually equal (so something like $\pi_Y^* \mathbf{R}\pi_{Y*}$ means to push down one of the projections, and pull along the other). Then we find that (being very explicit about the pushforward in particular):

$$\begin{aligned}
(\Phi_{\mathcal{K}}^{Y\to Z} \circ \Phi_{\mathcal{E}}^{X\to Y})(\mathcal{F}) &= \Phi_{\mathcal{K}}^{Y\to Z}(\mathbf{R}\pi_{Y*}^{G\times\mathbb{1}}(\pi_X^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{E})) \\
&= \mathbf{R}\pi_{Z*}^{H\times\mathbb{1}}(\pi_Y^*(\mathbf{R}\pi_{Y*}^{G\times\mathbb{1}}(\pi_X^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{E})) \overset{\mathbf{L}}{\otimes} \mathcal{K}) \\
&= \mathbf{R}\pi_{Z*}^{H\times\mathbb{1}}(\mathbf{R}\pi_{YZ*}^{G\times\mathbb{1}\times\mathbb{1}}(\pi_{XY}^*(\pi_X^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{E})) \overset{\mathbf{L}}{\otimes} \mathcal{K}) \\
&= \mathbf{R}(\pi_Z \circ \pi_{YZ})_*^{G\times H\times\mathbb{1}}((\pi_X \circ \pi_{XY})^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \pi_{XY}^*\mathcal{E}) \overset{\mathbf{L}}{\otimes} \pi_{YZ}^*\mathcal{K}) \\
&= \mathbf{R}(\pi_Z \circ \pi_{XZ})_*^{G\times H\times\mathbb{1}}((\pi_X \circ \pi_{XZ})^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \pi_{XY}^*\mathcal{E}) \overset{\mathbf{L}}{\otimes} \pi_{YZ}^*\mathcal{K}) \\
&= \mathbf{R}\pi_{Z*}^{G\times\mathbb{1}\times\mathbb{1}}(\pi_X^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathbf{R}\pi_{XZ*}^{\mathbb{1}\times H\times\mathbb{1}}(\pi_{XY}^*\mathcal{E} \overset{\mathbf{L}}{\otimes} \pi_{YZ}^*\mathcal{K})) \\
&= \mathbf{R}\pi_{Z*}^{G\times\mathbb{1}\times\mathbb{1}}(\pi_X^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{E} * \mathcal{K}) \\
&= \Phi_{\mathcal{E}*\mathcal{K}}^{X\to Z}.
\end{aligned}$$

Where we have used, in order, Proposition 5.6, Proposition 5.5, and Proposition 5.6 a second time. Thus we see that the convolution of the two kernels is $\mathcal{E} * \mathcal{K} = \mathbf{R}\pi_{XZ*}^{\mathbb{1}\times H\times\mathbb{1}}(\pi_{XY}^*\mathcal{E} \overset{\mathbf{L}}{\otimes} \pi_{YZ}^*\mathcal{K})$. ■

As before we can try to see what the identity functor is on the equivariant derived category when the action is trivial. In the ordinary setting, the kernel was $\mathcal{O}_\Delta$. However we need to tensor with something which preserves the equivariant structure. Recall that if G acts trivially on X (and G is finite), every equivariant sheaf $\mathcal{E}$ has a decomposition into simple objects

$$\mathcal{E} \cong \bigoplus \mathcal{E}_i \otimes \rho_i,$$

where the $\mathcal{E}_i$ are ordinary sheaves, and ρ_i the irreducible represenations of G. We claim that tensoring with $\mathcal{O}_X \otimes \mathbb{C}[G]$ is the identity functor on $\operatorname{Qcoh}^G X$. Indeed using the decomposition above yields

$$\mathcal{E} \otimes \mathcal{O}_X \otimes \mathbb{C}[G] = \bigoplus (\mathcal{E}_i \otimes \mathcal{O}_X) \otimes (\rho_i \otimes \mathbb{C}[G]) = \mathcal{E},$$

as since any representation is a $\mathbb{C}[G]$ module, tensoring with the sheaf determined by the group algebra acts as the identity. So we see that the identity functor on $\mathrm{D}_G^b(X)$ is isomorphic to the Fourier-Mukai transform $\Phi_{\mathcal{O}_\Delta \otimes \mathbb{C}[G]}^{X\to X}$, whose proof goes in much the same way as for the ordinary case. If G acts nontrivially on X, then this is no longer the correct kernel, rather something like $\oplus \mathcal{O}_{\Gamma_g}$ is the right one, but since we don't need this, we won't discuss it further.

6.0.1. *Geometric McKay Correspondence.* As an interesting aside, recall that the Grothendieck group of $\operatorname{Coh} X$, $K_0(\operatorname{Coh} X)$, is the free abelian group on elements of $\operatorname{Coh} X$, modulo the relation $\mathcal{F} = \mathcal{F}' + \mathcal{F}''$, where $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of coherent sheaves. Then given any complex F in $\mathrm{D}^b(X)$, we can assign to it an element of $K_0(\mathrm{Coh}\, X)$ by $[\mathcal{F}] = \sum(-1)^i[\mathcal{F}^i]$. Such an assignment turns a Fourier-Mukai transform $\Phi_{\mathcal{K}}^{X\to Y}$ to a K-theoretic integral transform. Indeed we simply define $f^*[\mathcal{F}] = [f^*\mathcal{F}]$ and $\mathbf{R}f_*[\mathcal{F}] = [\mathbf{R}f_*\mathcal{F}]$, and the product as $[\mathcal{F}][\mathcal{E}] = [\mathcal{F} \overset{L}{\otimes} \mathcal{E}]$. Making these replacements in the definition of a Fourier-Mukai transform then yields the following commutative diagram:

$$\begin{array}{ccc} \mathcal{D}^b(X) & \xrightarrow{\Phi_{\mathcal{K}}^{X\to Y}} & \mathcal{D}^b(Y) \\ {\scriptstyle [-]}\big\downarrow & & \big\downarrow{\scriptstyle [-]} \\ K_0(X) & \xrightarrow{[\Phi]_{[\mathcal{K}]}^{X\to Y}} & K_0(Y) \end{array}$$

Hence a Fourier-Mukai transform (that is, an equivalence) induces an isomorphism of Grothendieck groups (but the converse is not true). We will see in the proof of Theorem 1.1 that the equivalence is given by an equivariant Fourier-Mukai transform when associate the two categories $\mathrm{D}^b(X)$ and $\mathrm{D}^b_{\mathbb{1}}(X)$. Hence if we generalize the above diagram to the equivariant case, we see that

$$K_0(X) \cong K_0^G(\mathbb{C}^2),$$

which is the famous result of Gonzalez-Springberg and Verdier in [17], and is referred to as the *Geometric McKay correspondence.*

7. The Proof

We begin now with the hard part, proving Theorem 1.1. The resolution $c_G : X = \mathrm{Hilb}^G_{reg}\,\mathbb{C}^2 \to \mathbb{C}^2/G$ that we established in Theorem 3.5 comes with a universal family $\mathcal{Z} \subset X \times \mathbb{C}^2$ by the construction of the Hilbert scheme, so then we can form the commutative square

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{p_2} & \mathbb{C}^2 \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow[c_G]{} & \mathbb{C}^2/G \end{array}$$

where p_1, π are finite (hence proper), c_G, p_2 are birational, and p_1 is flat. Of course, G is acting on $\mathbb{C}^2$ via its tautological action, and so the idea is to endow X with a $\mathbb{1}$-action (which is trivial), which then induces a $\mathbb{1}\times G$-action on $\mathcal{Z} \subset X \times \mathbb{C}^2$. Then we use the commutative square to define an equivariant integral transform of equivariant derived categories. Further, we make the (slightly harmless) abuse of notation by writing $\mathrm{D}^b(X)$ for $\mathrm{D}^b_{\mathbb{1}}(X)$.

So toward this, we define two functors:

$$\Phi^{\mathbb{C}^2\to X}_{\mathcal{O}_{\mathcal{Z}}} : \mathrm{D}^b_G(\mathbb{C}^2) \to \mathrm{D}^b(X), \quad \mathcal{E} \mapsto \mathbf{R}p^G_{1*}(p_2^*\mathcal{E} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{\mathcal{Z}}),$$

where $\mathcal{O}_{\mathcal{Z}}$ is the pushforward of the structure sheaf of the universal family along the inclusion $i : \mathcal{Z} \to X \times \mathbb{C}^2$. In the other direction, we define the equivariant integral transform

$$\Phi^{X\to\mathbb{C}^2}_{\mathcal{O}^\vee_{\mathcal{Z}}} : \mathrm{D}^b(X) \to \mathrm{D}^b_G(\mathbb{C}^2), \quad \mathcal{F} \mapsto \mathbf{R}p^{\mathbb{1}}_{2*}(p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{O}^\vee_{\mathcal{Z}}),$$

where $\mathcal{O}_{\mathcal{Z}}^{\vee} = \mathbf{R}\mathcal{H}om(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{X\times\mathbb{C}^2})$ is the derived dual in $\mathrm{D}^b_{\mathbb{1}\times G}(X \times \mathbb{C}^2)$.

Proposition 7.1. *The two functors $\Phi_{\mathcal{O}_{\mathcal{Z}}}^{\mathbb{C}^2\to X}$ and $\Phi_{\mathcal{O}_{\mathcal{Z}}^{\vee}}^{X\to\mathbb{C}^2}$ are adjoint. That is,*

$$\mathcal{H}om_{\mathcal{O}_X}(\Phi_{\mathcal{O}_{\mathcal{Z}}}^{\mathbb{C}^2\to X}(\mathcal{F}), \mathcal{E}) \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^2}}(\mathcal{F}, \Phi_{\mathcal{O}_{\mathcal{Z}}^{\vee}}^{X\to\mathbb{C}^2}(\mathcal{E})).$$

Proof. Note that p_1 is proper, hence $\mathbf{R}p_{1*}$ has a right adjoint $p_1^!$ by Grothendieck-Verdier duality. Then the proof goes as Proposition 6.1, noting that the canonical sheaf is trivial. ■

Now if we consider the triple product $X \times \mathbb{C}^2 \times X$, and denote by $\mathcal{Z}^T \subset \mathbb{C}^2 \times X$ the transpose of $\mathcal{Z}$, and by $\mathcal{Z}_{12} = \pi_{12}^{-1}(\mathcal{Z})$, $\mathcal{Z}_{23}^T = \pi_{23}^{-1}(\mathcal{Z}^T)$, and in the triple product $\mathbb{C}^2 \times X \times \mathbb{C}^2$ with pairwise projections ρ_{ij} we set analogous notation, then we claim the following proposition.

Proposition 7.2. (1) *The composition $\Phi_{\mathcal{O}_{\mathcal{Z}}}^{\mathbb{C}^2\to X} \circ \Phi_{\mathcal{O}_{\mathcal{Z}}^{\vee}}^{X\to\mathbb{C}^2}$ has the kernel*

$$\mathcal{L} = \mathbf{R}\pi_{13*}(\mathbf{R}\mathcal{H}om(\mathcal{O}_{\mathcal{Z}_{12}}, \mathcal{O}_{\mathcal{Z}_{23}^T}))^G \in \mathrm{D}^b(X \times X).$$

(2) *The composition $\Phi_{\mathcal{O}_{\mathcal{Z}}^{\vee}}^{X\to\mathbb{C}^2} \circ \Phi_{\mathcal{O}_{\mathcal{Z}}}^{\mathbb{C}^2\to X}$ has the kernel*[11]*.*

$$\mathcal{M} = \mathbf{R}\rho_{13*}(\mathbf{R}\mathcal{H}om(\mathcal{O}_{\mathcal{Z}_{23}}, \mathcal{O}_{\mathcal{Z}_{12}^T})) \in \mathrm{D}^b_G(\mathbb{C}^2 \times \mathbb{C}^2).$$

Proof. Note that if f is a flat morphism, then $f^*\mathbf{R}\mathcal{H}om(E, F) = \mathbf{R}\mathcal{H}om(f^*E, f^*F)$. Now use this remark and Proposition 6.2. ■

Since the action on X is trivial, its enough to show that $\mathcal{L}$ is quasi-isomorphic to $\mathcal{O}_{\Delta_X}$, as then the composition of the integral transforms will be isomorphic (as functors) to the identity functor. Similarly, its enough to show $\mathcal{M}$ is quasi-isomorphic to $\mathcal{O}_{\Delta_{\mathbb{C}^2}} \otimes \mathbb{C}[G]$, where $\mathbb{C}[G]$ is the regular representation of G (recall from earlier what this notation means). Now let η be the tautological representation of G (as G is a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$). Now construct a Koszul resolution of the sheaf $\mathcal{O}_{\mathcal{Z}}$, by first resolving $\mathcal{O}_\xi$ for $\xi \in X$. This yields a Koszul resolution

$$[p_1^*\mathcal{E} \to (p_1^*\mathcal{E})^{\oplus 2} \to p_2^*\mathcal{E}] \cong \mathcal{O}_{\mathcal{Z}}$$

which we denote by $\mathcal{K}$.

Thus we see by the equivariant projection formula:

$$\begin{aligned}\Phi_{\mathcal{O}_{\mathcal{Z}}}^{\mathbb{C}^2\to X}(\mathcal{F}) &= \mathbf{R}p_{1*}^G(p_2^*\mathcal{E} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{\mathcal{Z}}) \\ &= ((\mathbf{R}p_{1*}p_2^*\mathcal{F}) \otimes \mathcal{E})^G \to ((\mathbf{R}p_{1*}p_2^*\mathcal{F}) \otimes \mathcal{E}^{\oplus 2})^G \to ((\mathbf{R}p_{1*}p_2^*\mathcal{F}) \otimes \mathcal{E})^G\end{aligned}$$

is the image of $\mathcal{F}$ under $\Phi_{\mathcal{O}_{\mathcal{Z}}}^{\mathbb{C}^2\to X}(\mathcal{F})$. Note that $\mathbf{R}p_{1*}p_2^*\mathcal{F}$ is simply $\Gamma(\mathbb{C}^2, \mathcal{F})$, as the higher cohomology of $\mathbb{C}^2$ vanishes. Since $\mathcal{F}$ is also a G-equivariant sheaf, this splits into irreducible representations via Maschke's theorem, and so

$$\Phi_{\mathcal{O}_{\mathcal{Z}}}^{\mathbb{C}^2\to X}(\mathcal{F}) = \oplus_{\rho_i}\Gamma(\mathbb{C}^2, \mathcal{F})_{\rho_i} \otimes \mathcal{E}_{\rho_i} \to \oplus_{\rho_i,\rho_j}\Gamma(\mathbb{C}^2, \mathcal{F})_{\rho_i,\rho_j} \otimes \mathcal{E}_{\rho_i}^{m_{ij}} \to \oplus_{\rho_i}\Gamma(\mathbb{C}^2, \mathcal{F})_{\rho_i} \otimes \mathcal{E}_{\rho_i}$$

where m_{ij} is the multiplicity of ρ_j in $\rho_i \otimes \eta$.

[11] We would like to point out a typo in the original paper of [27]. They instead have $\mathbf{R}\rho_{13*}(\mathbf{R}\mathcal{H}om(\mathcal{O}_{\mathcal{Z}_{12}^T}, \mathcal{O}_{\mathcal{Z}_{23}}))$, which is not correct.

By a completely analogous argument, we get

$$\Phi_{\mathcal{O}_{\mathcal{Z}}^{\vee}}^{X\to\mathbb{C}^2}(\mathcal{G}) = \mathbf{R}\Gamma(X,\mathcal{G}\otimes\mathcal{E}^{\vee})\otimes\mathcal{O}_{\mathbb{C}^2} \to \mathbf{R}\Gamma(X,\mathcal{G}\otimes\mathcal{E}^{\vee})\otimes\mathcal{O}_{\mathbb{C}^2}^{\oplus 2} \to \mathbf{R}\Gamma(X,\mathcal{G}\otimes\mathcal{E}^{\vee})\otimes\mathcal{O}_{\mathbb{C}^2}$$

and here $\mathcal{O}_{\mathbb{C}^2}$ carries the equivariant structure determined by η.

Now we see that $\mathcal{L}$ is quasi-isomorphic to the complex given by

$$\left\{\bigoplus_{\rho} E_{\rho}^{\vee}\boxtimes\mathcal{E}_{\rho} \to \bigoplus_{\rho_i,\rho_j} E_{\rho_i}^{\vee}\boxtimes\mathcal{E}_{\rho_j}^{m_{ij}} \to \bigoplus_{\rho} E_{\rho}^{\vee}\boxtimes\mathcal{E}_{\rho}\right\}$$

where $\boxtimes$ indicates the external tensor. Now the proof that this complex is quasi-isomorphic to $\mathcal{O}_{\Delta}$ is done in [37].

All we have left is to examine $\mathcal{M}$.

References

[1] Claudio Bartocci, Ugo Bruzzo, Daniel Hernández Ruipérez. *Fourier-Mukai and Nahm Transforms in Geometry and Mathematical Physics*. Progress in Mathematics, No. 276, Birkhaüser, Boston, 2009.

[2] Joseph Bernstein, Valery Lunts. *Equivariant Sheaves and Functors*. Lecture Notes in Mathematics, No. 1578, Springer-Verlag, Berlin, 1994.

[3] T. Bridgeland, A. King, M. Reid, *The McKay correspondence as an equivalence of derived categories*. J. Amer. Math. Soc. 14 (2001), 535–554.

[4] Winfried Bruns, Jürgen Herzog. *Cohen-Macaulay Rings*, revised edition. Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1998.

[5] Daniel Chan, Colin Ingalls. *Non-commutative Coordiante Ring an Stacks*. Proc. London Math. Soc. (3) 88 (2004) 63-88.

[6] Neil Chriss, Victor Ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhaüser Classics, Birkhaüser, Boston, 1997.

[7]

[8] Igor Dolgachev. *McKay Correspondence*. Lecture Notes, http://www.math.lsa.umich.edu/~idolga/McKaybook.pdf

[9] Igor Dolgachev. *Derived Categories*. Lecture Notes, http://www.math.lsa.umich.edu/~idolga/derived9.pdf

[10]

[11]

[12] David Eisenbud. *Commutative Algebra: With a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, No. 150, Springer-Verlag, New York, 1995.

[13]

[14] John Fogarty. *Algebraic families on an algebraic surface*. Amer. J. Math, 90:511-521, 1968.

[15]

[16] Alexander Grothendieck. *Sur quelques points d'algèbre homologique*, Tôhoku Math. J., (1957), pp. 119-221.

[17] G. Gonzalez-Springberg, J.L. Verdier, *Construction géométrique de la correspondance de McKay*, Ann. ENS, 16 (1983), 409-449.

[18] Mark Haiman. *t, q-Catalan numbers and the Hilbert scheme*. Discrete Math., 193(1-3):201-224, 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).

[19]

[20] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York, 1977.

[21] Robin Hartshorne. *Residues and Duality*. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966.

[22] Robin Hartshorne. *Stable reflexive sheaves*, Math. Ann. 254 (1980), no. 2, 121–176.
[23] Daniel Huybrechts. *Fourier-Mukai Transforms in Algebraic Geometry.* Oxford Mathematical Monographs, Oxford Science Publications, New York, 2006.
[24] Hochster, M. and J.L. Roberts. *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. 13 (1974), 115-175.
[25] Shihoko Ishii. *Introduction to Singularities*, second ed., Springer Japan, Springer, 2014.
[26] Y. Ito and I. Nakamura. *Hilbert schemes and simple singularities.* In New trends in algebraic geometry (Warwick, 1996), volume 264 of London Math. Soc. Lecture Note Ser., pages 151-233. Cambridge Univ. Press, Cambridge, 1999.
[27] M. Kapranov and E. Vasserot, *Kleinian singularities, derived categories, and Hall algebras*, Math. Ann. 316 (2000), 565-576.
[28] Alexander Kirillov Jr. *Quiver Representations and Quiver Varieties.* Graduate Studies in Mathematics, vol. 174. American Mathematical Society, Providence, RI, 2016.
[29]
[30]
[31] Joseph Lipman, *Rational singularities, with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 195–279.
[32]
[33]
[34] Hideyuki Matsumura. *Commutative Ring Sheory*, second ed. Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989.
[35]
[36] add citation
[37] Hiraku Nakajima. *Lectures on Hilbert schemes of points on surfaces.* University Lecture Series, No. 18, American Mathematical Society, Providence, RI, 1999.
[38]
[39]
[40] add citation
[41] Zhenbo Qin. *Hilbert schemes of points and infinite dimensional Lie algebras.* Mathematical Surveys and Monographs, vol. 228. American Mathematical Society, Providence, RI, 2018.
[42] Miles Reid. *The Du Val Singularities* A_n*,* D_n*,* E_6*,* E_7*,* E_8. Lecture Notes, https://homepages.warwick.ac.uk/~masda/surf/more/DuVal.pdf
[43] Miles Reid. *Young person's guide to canonical singularities.* In Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), volume 46 of Proc. Sympos. Pure Math., pages 345-414. Amer. Math. Soc., Providence, RI, 1987.
[44] Karl Schwede. *Generalized Divisors and Reflexive Sheaves.* Lecture Notes, https://www.math.utah.edu/~schwede/Notes/GeneralizedDivisors.pdf
[45] Verdier, Jean-Louis (1996). *Des Catégories Dérivées des Catégories Abéliennes.* Astérisque (in French). Société Mathématique de France, Marseilles. 239.
[46] add citation