
The Calabi Conjecture and Complex Monge-Ampère Equations

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1 Kähler Geometry

Kähler manifolds serve as a very rich class of objects to study in many areas of math, and in these notes we will explore a specific part of this area. Typically Kähler manifolds show up in areas of math involving Algebraic and Differential Geometry, and here we will discuss how analysis of several complex variables contributes to this subject. Throughout, we will work exclusively with *compact* Kähler manifolds.

1.1 Introduction, Basic Definitions

We begin with the definition of a compact complex manifold, and give a few examples.

Definition 1.1 (Complex Manifold). *Let X be a compact Hausdorff topological space. A system of local coordinates on X is a cover $\{U_\alpha\}$ with homeomorphisms $z_\alpha : U_\alpha \rightarrow \mathbb{C}^n$, such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the composition*

$$\phi_{\alpha\beta} = z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \rightarrow z_\beta(U_\alpha \cap U_\beta)$$

is a biholomorphic map.

We say that X is a compact complex manifold of dimension n , if it can be endowed with a complex structure, i.e. an equivalence class of systems of local coordinates (where two systems $\{(z_\alpha, U_\alpha)\}, \{(w_\beta, V_\beta)\}$ are equivalent if the maps $z_\alpha(p) \rightarrow w_\beta(p)$ are biholomorphic whenever defined).

In particular, a complex manifold of dimension n can be regarded as a smooth (real) manifold of dimension $2n$. There are several notions of the tangent space to a point $p \in X$:

1. the usual real tangent space $T_{\mathbb{R},p}X$, which can be thought of as the space of real derivations on the ring of C^∞ functions near p (or equivalently, the stalk at p of the sheaf of derivations acting on the sheaf of smooth functions on X);
2. the complexified tangent space $T_{\mathbb{C},p}X = T_{\mathbb{R},p}X \otimes \mathbb{C}$;
3. the holomorphic tangent space $T'_pX \subset T_{\mathbb{C},p}X$ consisting of derivations that vanish on anti-holomorphic sections.

If we have a complex manifold X , as before we can regard it as a real $2n$ -dimensional smooth manifold, equipped with extra data. This extra data consists of an endomorphism J of the real tangent bundle $T_{\mathbb{R}}X$ satisfying

$$\begin{cases} J^2 = -\text{id} \\ J(\partial_{x_j}) = \partial_{y_j} \\ J(\partial_{y_j}) = -\partial_{x_j}. \end{cases}$$

It is worth remarking that the holomorphic tangent bundle is the bundle of eigenspace corresponding to i when J acts on $T_{\mathbb{C}}X$. In addition, recall that a Riemannian metric g smoothly assigns an inner product $\langle \cdot, \cdot \rangle_p$ to each T_pX .

Definition 1.2 (Hermitian Metric). *A Riemannian metric on X is called Hermitian if it satisfies*

$$g(u, v) = g(Ju, Jv)$$

for all $u, v \in T_p X$ and all $p \in X$.

It should be clear that we can write a Hermitian metric as

$$g = 2\Re \left(\sum_{\alpha, \beta} g_{\alpha\beta} dz_\alpha \otimes d\bar{z}_\beta \right)$$

where the matrix $(g_{\alpha\beta})$ is a Hermitian matrix. From this matrix, we can associate a form of bidegree (1,1), called the fundamental form:

$$\omega = \sum_{\alpha, \beta} g_{\alpha\beta} i dz_\alpha \wedge d\bar{z}_\beta.$$

Definition 1.3. *We say that a Hermitian manifold is Kähler if its fundamental form ω is closed. In this case we call the fundamental form the Kähler form.*

1.2 Line Bundles, Chern Classes

We will now change gears to discuss a more topological construction, the Chern class of a line bundle. We have already implicitly assumed that the reader knows the definition of a vector bundle, but we will repeat it here in the case of line bundles.

Definition 1.4. *Let X be a complex manifold. A holomorphic line bundle $L \xrightarrow{\pi} X$ is a complex manifold such that*

1. for each $x \in X$, $\pi^{-1}(x)$ is a one dimensional complex vector space;
2. the projection map $\pi : L \rightarrow X$ is holomorphic;
3. for each $x \in X$, there exists an open neighborhood $U \subset X$ of x and

$$\phi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{C}$$

is a biholomorphic map.

The map ϕ_U is called a local trivialization of L . Note that for any pair of trivializations ϕ_U, ϕ_V , there is a holomorphic map $g_{UV} \in \mathcal{O}(U \cap V, \mathbb{C}^*)$ such that $\phi_U \circ \phi_V^{-1}$ maps $U \cap V \times \mathbb{C}$ to itself via

$$(x, v) \mapsto (x, g_{UV}(v)).$$

These maps g_{UV} are called the cocycles or transition maps for the line bundle, and in particular they satisfy

$$g_{UV} \cdot g_{VU} = 1, \quad g_{UV} \cdot g_{VW} \cdot g_{WU} = 1.$$

These are called the cocycle conditions.

Conversely, given a cover and a collection of transition maps, there exists a unique (up to isomorphism) line bundle having those maps as its transition functions (this proof is not too difficult, so we again leave to the reader). The cocycle conditions on the above maps turn out to exactly be the requirements for $\{g_{UV}\}$ to be a Čech 1-cochain on X with values in the sheaf \mathcal{O}^* . Thus we see that

the Čech cohomology group $H^1(X, \mathcal{O}^*)$ parametrizes the line bundles on X up to isomorphism. This group is referred to as the Picard group of X .

Now we have the so-called exponential exact sequence on sheaves:

$$0 \rightarrow \mathbb{Z} \xrightarrow{2i\pi(\cdot)} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0$$

which induces a long exact sequence in cohomology, of which there is a boundary map

$$\delta_1 : H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathbb{Z}).$$

Definition 1.5. *Let L be a line bundle. This line bundle determines, up to isomorphism, a cohomology class $[L] \in H^1(X, \mathcal{O}^*)$, and we define the first Chern class of L , $c_1(L)$ to be $\delta_1([L])$.*

We will sometimes be interested in the image of $c_1(L)$ under the map $H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{R})$ induce by the normal inclusion.

There is very important line bundle in particular that we are interested in, namely the canonical bundle.

Definition 1.6. *Let $\Omega_{\mathbb{C}}X = (T_{\mathbb{C}}X)^* = \Omega^{1,0}X \oplus \Omega^{0,1}X$, and we set the canonical bundle to be $K_X = \wedge^n \Omega^{1,0}X = \Omega^{n,0}X$. The anticanonical bundle is the bundle K_X^* .*

We then, for any complex manifold X , say that the first Chern class of X is the first Chern class associated to the anticanonical bundle of X , $c_1(X) = c_1(K_X^*)$. This is (of course) a cohomology class that does not depend on ω , but rather on the complex structure on M .

There is one more result that we will use below. The reader has likely heard of it (at the very least!), but we put it here for completeness.

Lemma 1.0.1 ($\partial\bar{\partial}$ -lemma). *Let X be a compact Kähler manifold. Let S be a current which is both ∂ and $\bar{\partial}$ closed. Then S is d exact if and only if it is dd^c exact.*

1.3 Curvature

As you may recall, on a Riemannian manifold with metric g there is a unique connection ∇ , called the Levi-Civita connection, which satisfies:

1. $\nabla g = 0$;
2. $[X, Y] = \nabla_X Y - \nabla_Y X$, for any two vector fields X, Y .

Similarly, on any Hermitian vector bundle, there is a canonical “nice” connection, called the Chern connection $\hat{\nabla}$ that satisfies very similar properties as above. Namely:

1. $\hat{\nabla} g = 0$;
2. $\hat{\nabla} J = 0$;
3. the torsion is pure in its indices.

When the base manifold is a Kähler manifold and the vector bundle is its tangent bundle, these two connections agree. We can look at the Christoffel symbols on $T_{\mathbb{C}}X$:

$$\nabla_{z_i} \partial_{z_j} = \sum_l \Gamma_{ij}^l \partial_{z_l} + \sum_k \Gamma_{ij}^{\bar{k}} \partial_{\bar{z}_k}$$

and

$$\nabla_{z_i} \partial_{\bar{z}_j} = \sum_l \Gamma_{i\bar{j}}^l \partial_{z_l} + \sum_k \Gamma_{i\bar{j}}^{\bar{k}} \partial_{\bar{z}_k}.$$

An application of the Kähler condition forces the only nonzero terms to be $\Gamma_{i\bar{j}}^l$ and $\Gamma_{i\bar{j}}^{\bar{k}}$. We can similarly define the Riemann curvature tensor as in Riemannian geometry, except due to the restrictions above on nonzero Christoffel symbols, we find that the only nonzero terms are the coefficients of the form

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{ij}}{\partial z_k \partial \bar{z}_l} + \sum_{s,t} g^{st} \frac{\partial g_{sj}}{\partial z_k} \frac{\partial g_{it}}{\partial \bar{z}_l}.$$

Again as in Riemannian geometry, the often easier quantity to work with is the Ricci curvature tensor, which is defined as the trace of the Riemann curvature tensor:

$$\text{Ric}_{kl} = \sum_{i,j} g^{ij} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_l} \log \det(g_{pq})$$

and using the Kähler form, we can also associate the Ricci form to this:

$$\text{Ric}(\omega) = \frac{ic}{2} \sum_{\alpha,\beta} \text{Ric}_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta.$$

A fundamental consequence of this is if $\tilde{\omega}$ is another Kähler form on X , then $\text{Ric}(\tilde{\omega}) - \text{Ric}(\omega) = \frac{1}{2} dd^c \log \frac{\tilde{\omega}^n}{\omega^n}$.

Lemma 1.0.2. *The Ricci form is a closed (1,1)-form representing the first Chern class $c_1(X)$ of X .*

This lemma now brings us to the Calabi conjecture.

1.4 The Calabi Conjecture

Given a compact Kähler manifold we have already seen that the Ricci form $\text{Ric}(\omega)$ is a closed (1,1) form representing the first Chern class of X . Calabi, in the 50's, asked the converse. That is, given a closed (1,1) form η representing $c_1(X)$, and $\alpha \in H^{1,1}(X, \mathbb{R})$ a Kähler class, whether or not there exists a Kähler form ω in α such that $\text{Ric}(\omega) = \eta$. The answer to this question is yes, and it was proven by Yau in 1978.

Theorem 1.1. *Let X be a compact Kähler manifold and fix a Kähler class $[\alpha] \in H^{1,1}(X, \mathbb{R})$. Given a smooth closed (1,1)-form η representing $c_1(X)$, there exists a unique Kähler form $\omega \in [\alpha]$ such that $\text{Ric}(\omega) = \eta$.*

Our goal from now on will be to prove (most of) this theorem.

2 Strategy for the Proof

2.1 Reformulation of the Conjecture

Fix η as a representative of $c_1(X)$ and ω a Kähler form in a given Kähler class. Since $\text{Ric}(\omega)$ also represents $c_1(X)$, it follows from the $\partial\bar{\partial}$ -lemma that there exists $h \in C^\infty(X, \mathbb{R})$ such that

$$\text{Ric}(\omega) = \eta + dd^c h.$$

We now seek a new Kähler form, cohomologous to ω , so that $\omega_\varphi = \omega + dd^c\varphi$, satisfying $\text{Ric}(\omega_\varphi) = \eta$. So we are attempting to solve

$$\text{Ric}(\omega_\varphi) - \text{Ric}(\omega) - dd^c h = dd^c \left(\log \left\{ \frac{(\omega + dd^c\varphi)^n}{\omega^n} \right\} - h \right) = 0.$$

However this implies that the argument is pluriharmonic, and so on X , which is compact, it must be constant. So shifting h by a constant if necessary, we see by taking the exponential of the resulting equation that we must solve

$$(\omega + dd^c\varphi)^n = e^h \omega^n,$$

which is a nonlinear inhomogeneous PDE. Note that we require h also satisfies the normalizing condition:

$$\int_X e^h \omega^n = \int_X \omega^n = \text{Vol}(X).$$

2.2 How Do We Solve It?

Our strategy is to use the continuity method. That is, we consider a smoothly varying family of equations

$$(\omega + dd^c\varphi)^n = [te^h + (1-t)]\omega^n, \tag{Y_t}$$

each satisfying the normalizing condition as above. Clearly, when $t = 0$, the equation admits the trivial solution $\varphi \equiv 0$, so it is nonempty, while $t = 1$ is the case that we are interested in. So our strategy is, fixing $k \in \mathbb{N}, 0 < \alpha < 1$, to now show that the set

$$\Sigma = \{s \in [0, 1] \mid (Y_s) \text{ has a smooth solution } \varphi_s \in PSH(X, \omega) \cap C^{k+2, \alpha}(X)\}$$

is open and closed in $[0, 1]$, then by connectedness, we are done. To take care of the normalizing condition, we also require $\int_X \varphi_s \omega^n = 0$.

3 A Sketch of the Proof

3.1 Preliminaries

We will from now on that ω is a fundamental form for our compact Kähler manifold X , and further:

$$\int_X \omega^n = 1.$$

In addition we will adopt the notation ω_φ used previously to mean $\omega_\varphi = \omega + dd^c\varphi$. We will call a continuous function ω -plurisubharmonic (ω -psh) if $\omega_\varphi \geq 0$. These functions are needed as we have no global plurisubharmonic functions on X due to the maximum principle.

If on some open subset of X there is a function g such that $\omega = dd^c g$, then $\varphi + g$ is a true plurisubharmonic function, so many properties of plurisubharmonic functions also hold for ω -psh functions. One particularly useful property is that the regularization of a limit of a uniformly bounded sequence is also ω -psh.

For a Borel set $E \subset X$, define the capacity:

$$\text{cap}_\omega(E) = \sup \left\{ \int_X \omega_\varphi^n \mid \varphi \in PSH(\omega), 0 \leq \omega \leq 1 \right\}.$$

A sequence φ_j of functions defined on X is said to converge with respect to the capacity if for any $t > 0$,

$$\lim_{j \rightarrow \infty} \text{cap}_\omega(\{|\varphi - \varphi_j| \geq t\}) = 0.$$

We have an analogue of the comparison principle for this case as well:

Theorem 3.1. *If φ and ψ are ω -psh on X , then on $\Omega = \{\varphi < \psi\}$ we have*

$$\int_{\Omega} \omega_{\psi}^n \leq \int_{\Omega} \omega_{\varphi}^n.$$

The proof is similar to the normal case. There is also an important estimate due to Hörmander, called the exponential estimate, which can be extended to Kähler manifolds in the following way:

Theorem 3.2. *Let ω be a Kähler form. There exists a strictly positive number α and a constant C depending only on ω so that*

$$\frac{1}{\int_X \omega^n} \int_X e^{-\alpha(u - \sup_X u)} \omega^n \leq C$$

for all $u \in PSH(X, \omega)$.

3.2 Uniqueness

We first show uniqueness of this equation. Existence is much tougher and we will tackle it later.

Proposition 3.1. *Assume that X has no boundary. If φ and ψ are continuous on X and $\omega_{\varphi} > 0$, $\omega_{\psi} > 0$ and $\omega_{\varphi}^n = \omega_{\psi}^n$, then $\varphi - \psi$ is constant.*

As a note, the assumption that $\omega_{\varphi} > 0$ and $\omega_{\psi} > 0$ is quite reasonable, as we are implicitly assuming that ω is a Kähler form. In addition the normalizing condition we have enforced is now motivated; by enforcing functions with zero average, this forces actual uniqueness.

Proof. We will prove this in the case of $n = 2$, the general case is similar. We know already that

$$\omega_{\varphi}^2 - \omega_{\psi}^2 = (\omega + dd^c \varphi)^2 - (\omega + dd^c \psi)^2 = 0.$$

We can write this as $0 = dd^c(\varphi - \psi) \wedge (\omega_{\varphi} + \omega_{\psi})$, so set $\rho = \varphi - \psi$. Since $\omega_{\varphi} > 0$ and $\omega_{\psi} > 0$ in the sense of currents, we find that $\varphi - \psi$ is constant. ■

It is possible to generalize this proof to the case where $\omega_{\varphi} \geq 0$ and $\omega_{\psi} \geq 0$ (the degenerate case), but it is slightly more involved.

3.3 Existence

Recall the definition of the set Σ given earlier, we have our first proposition, which we will sketch instead of prove in painstaking detail.

Proposition 3.2. *The set Σ is open.*

Proof. Suppose $s \in \Sigma$, we must show that for t close to s , $t \in \Sigma$. Set $\omega_s = \omega + dd^c\varphi_s$, and $g_s = \log[se^h + (1-s)]$. This allows us to rewrite the problem as

$$\omega_s^n = [se^h + (1-s)]\omega^n = e^{g_s}\omega^n.$$

We are looking for $\omega_t = \omega + dd^c\varphi_t$ such that $\omega_t^n = e^{g_t}\omega^n$. Taking the difference of φ_t and φ_s , we find

$$(\omega_s + dd^c(\varphi_t - \varphi_s))^n = e^{g_t - g_s}\omega^n.$$

Call this problem (Y_{t-s}^*)

Now for $\epsilon = t - s$, observe that we can bound the right hand side:

$$\begin{aligned} k_\epsilon &= g_t - g_s \\ &= \log \left[\frac{te^h + (1-t)}{se^h + (1-s)} \right] \\ &= \log \left[1 + \epsilon \frac{e^h - 1}{se^h + (1-s)} \right] \\ &= O(\epsilon). \end{aligned}$$

Now (Y_{t-s}^*) is a problem “nearby” (Y_0) , which we already know has a smooth solution. A computation reveals that the differential of the map taking $\varphi \mapsto \omega_\varphi^n/\omega^n$ is the Laplacian, and so by some standard theory we now apply the inverse function theorem in an appropriate Banach space (the details of which we will skip), and we are done (note that this also takes care of the normalizing condition). ■

The proof to show that Σ is closed is much tougher. We need to establish several estimates to do this. Note that it suffices to show that we can find $\beta > \alpha$ and $C > 0$ such that for all $s \in \Sigma$,

$$\|\varphi_s\|_{C^{k+2,\beta}} \leq C. \tag{*}$$

Then we can use the fact that this family is uniformly bounded in $C^{k+2,\beta}$, and thus relatively compact in $C^{k+2,\beta}$. Indeed the family is actually compact as each limit point $\psi = \lim \varphi_{s_i}$, $s_i \rightarrow s$, is still normalized

$$\int_X \psi \omega^n = 0,$$

hence $\psi = \varphi_s$. However since we chose α arbitrarily, we can choose β arbitrarily, and since we will do this for any k , we will obtain a smooth solution. The general steps go as follows:

1. We will show that the family $\{\varphi_s\}$ is uniformly bounded in $L^\infty(X)$:

$$\|\varphi_s\|_{L^\infty(X)} \leq C.$$

2. Then (using the uniform bound above) we move on to show that there exists a $C_2 > 0$ such that for all $s \in \Sigma$,

$$\sup_X |\Delta\varphi_s| \leq C_2,$$

where $\Delta\varphi_s$ is the Laplace operator corresponding to the Kähler form ω . Then prove \star in the case of $k = 0$, and some $\beta > 0$

3. At this stage we can invoke standard estimates from the theory of elliptic equations, and use a bootstrapping argument to conclude the result.
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All but the first step will be left for other treatments, and our presentation stems from the pluripotential theoretic proof due to Kołodziej, adapted in [K,GZ]. Our line of reasoning will follow closely the survey [PSS]. In fact, we will prove a more general scenario, where the form is merely semi-positive and closed (as well as not uniformly zero). Set $\omega_t = \chi + (1-t)\omega$, where χ can be semi-positive and closed and ω is a Kähler form. As before consider the equation

$$(\omega_t + dd^c \varphi_t)^n = g_t \omega_t^n$$

for some positive function g_t and $\varphi_t \in PSH(X, \omega) \cap L^\infty(X)$. Then we have the following uniform estimate:

Theorem 3.3. *Let $A > 0$ and $p > 1$. Assume that $\chi \leq A\omega$ and $\frac{1}{\int_X \omega_t^n} \frac{\omega_t^n}{\omega^n} \leq A$. Assume also that $f_t \in L^p(X, \omega_t^n)$ and*

$$\frac{1}{\int_X \omega_t^n} \int_X f_t^p \omega_t^n \leq A^p < \infty$$

for all $t \in (0, 1)$. Normalize φ_t such that $\max_X \varphi_t = 0$. Then there exists a constant $C > 0$, depending only on n, ω and A , such that

$$\sup_{t \in [0, 1]} \|\varphi_t\|_{L^\infty(X)} \leq C.$$

Proof. Set

$$f_t(s) = \left(\frac{\text{cap}_\omega(\{\varphi_t < s\})}{\int_X \omega_t^n} \right)^{1/n}.$$

We claim that it suffices to show that there exists $s_m < \infty$ such that $f_t(s) = 0$ for all $s > s_m$. For convenience set $\{\omega_t^n\} = \int_X \omega_t^n$; by definition of the capacity, we know that $f_t(s)^n \geq \frac{1}{\{\omega_t^n\}} \int_{\varphi < -s} \omega_t^n$. If this were zero for all $s > s_m$, then we would have that $\omega_t \geq -s_m$ a.e. with respect to the measure ω_t^n , and since it is upper-semicontinuous, it satisfies the inequality everywhere. ■

So it remains to prove that there exists a finite s_m such that $f_t(s) = 0$ for all $s > s_m$. A lemma of De Giorgi will prove this, it states:

Lemma 3.3.1. *Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfy:*

1. *f is right-continuous;*
2. *f decreases to zero;*
3. *there exists positive constants α, A_α for which all $s \geq 0$ and all $0 \leq r \leq 1$, $rf(s+r) \leq A_\alpha f(s)^{1+\alpha}$.*

Then there exists s_m , depending only on the above constants and the smallest value s_0 for which $f(s_0)^\alpha \leq (2A_\alpha)^{-1}$ so that $f(s) = 0$ for $s > s_m$. Indeed we can take $s_m = s_0 + 2A_\alpha(1-2^{-\alpha})^{-1} f(s_0)^\alpha$.

We will show piece by piece that f_t satisfies the above lemma. The first is actually simple, as for any Kähler form ω , and any sequence of Borel sets $E_j \subset E_{j+1}$, we have $\text{cap}_\omega(\cup_j E_j) = \lim_{j \rightarrow \infty} \text{cap}_\omega(E_j)$. Now observe that as s approaches from the right, we get such a sequence of Borel sets.

It is clear that $f_t(s)$ decreases with s increasing. In fact, more is true, $f_t(s)$ decreases uniformly to zero as s increases.

Lemma 3.3.2. *There exists a constant C depending only on ω and an upper bound A for χ so that $f_t(s)^n \leq Cs^{-1}$.*

Proof. Let $u \in PSH(X, \omega_t)$. Then

$$\begin{aligned} \int_{\varphi_t < -s} (\omega_t + dd^c u)^n &\leq \frac{1}{s} \int_X (-\varphi_t)(\omega_t + dd^c u)^n \\ &= \frac{1}{s} \int_X (-\varphi_t)\omega_t^n + \frac{1}{s} \int_X [\text{Bonus Terms}], \end{aligned}$$

where the extra terms come from actually expanding out $(\omega_t + dd^c u)^n$ and collecting all terms other than ω_t^n . The first integral, using that $\omega_t^n \leq A\{\omega_t^n\}\omega^n$, and noting that by the definition of ω -psh functions, $PSH(X, \omega_t) \subset PSH(X, (A+1)\omega)$, we can use exponential boundedness to conclude that the first integral is bounded by $C\{\omega_t^n\}$. The second term, after a re-writing, can be bounded by a uniform constant times $\{\omega_t^n\}$ (the details are skipped for clarity). This establishes the claim. \blacksquare

It remains to establish the last condition for $f_t(s)$. For this, we need the following two results.

Lemma 3.3.3. *Let $\varphi \in PSH(X, \omega) \cap L^\infty(X)$. Then for all $s > 0$, $0 \leq r \leq 1$,*

$$r^n \text{cap}_\omega(\varphi < -s - r) \leq \int_{\varphi < -s} (\omega + dd^c \varphi)^n.$$

Lemma 3.3.4. *There exists constants $\delta, C > 0$ so that for any open set $E \subset X$, and any $t \in [0, 1)$, we have*

$$\frac{1}{\{\omega_t^n\}} \int_E \omega_t^n \leq C \exp \left[-\delta \left(\frac{\{\omega_t^n\}}{\text{cap}_\omega(E)} \right)^{1/n} \right].$$

If we assume these two lemmas for now, we can establish the final inequality for $f_t(s)$. For some $\alpha > 0$ we have

$$\begin{aligned} [r f_t(r+s)]^n &= r^n \frac{\text{cap}_{\omega_t}(\varphi_t < -s-r)}{\{\omega_t^n\}} \\ &\leq \frac{1}{\{\omega_t^n\}} \int_{\varphi_t < -s} (\omega_t + dd^c \varphi_t)^n \\ &= \frac{1}{\{\omega_t^n\}} \int_{\varphi_t < -s} g_t \omega_t^n \\ &\leq \left(\frac{1}{\{\omega_t^n\}} \int_{\varphi_t < -s} g_t^p \omega_t^n \right)^{1/p} \left(\frac{1}{\{\omega_t^n\}} \int_{\varphi_t < -s} \omega_t^n \right)^{1/q} \\ &\leq A \exp \left[-\frac{\delta}{q} \left(\frac{\{\omega_t^n\}}{\text{cap}_\omega(\varphi_t < -s)} \right)^{1/n} \right] \\ &\leq A_\alpha f_t(s)^{(1+\alpha)n}. \end{aligned}$$

Now we prove the two remaining lemmas.

Proof of Lemma 3.3.3. Let $u \in PSH(X, \omega)$ with $0 \leq u \leq 1$, and write

$$r^n \int_{\varphi < -s-r} (\omega + dd^c u)^n = \int_{\varphi < -s-r} (r\omega + dd^c ru)^n$$

$$\begin{aligned} &\leq \int_{\varphi < -s-r+ru} (\omega + dd^c(ru - s - r))^n \\ &\leq \int_{\varphi < -s-r+ru} (\omega + dd^c\varphi)^n \end{aligned}$$

where we applied the comparison principle. For the last integral, since $-r + ru$ is non-positive, this last integral is bounded by the integral over the larger region $\{\varphi < -s\}$, and taking the supremum, this proves the claim. \blacksquare

The next lemma requires some notions about global extremal functions.

Lemma 3.3.5. *Let $E \subset X$ be an open set, and define its global extremal function $\psi_{E,\omega}$ as the upper semi-continuous envelope of the following function*

$$\tilde{\psi}_{E,\omega} = \sup\{u \in PSH(X, \omega) \mid u = 0 \text{ on } E\}.$$

Then

1. $\psi_{E,\omega} \in PSH(X, \omega) \cap L^\infty(X)$;
2. $\psi_{E,\omega} = 0$ on E ;
3. $(\omega + dd^c\psi_{E,\omega})^n = 0$ on $X \setminus \bar{E}$.

Taking the above on faith (indeed, the proof follows from reducing it to a local statement about plurisubharmonic functions on \mathbb{C}^n , which in turn follows in much the same way as subharmonic functions on \mathbb{C}), we can now prove the final lemma.

Proof of Lemma 3.3.4. Let $E' \subset E$ be any relatively compact open subset. Then we see

$$\frac{1}{\{\omega_t^n\}} \int_{E'} \omega_t^n = \frac{1}{\{\omega_t^n\}} e^{-\delta \sup_X \psi_{E',\omega_t}} \int_{E'} e^{-\delta(\psi_{E',\omega_t} - \sup_X \psi_{E',\omega_t})} \omega_t^n$$

and this is bounded above by

$$e^{-\delta \sup_X \psi_{E',\omega_t}} A \int_X e^{-\delta(\psi_{E',\omega_t} - \sup_X \psi_{E',\omega_t})} \omega^n$$

where A is the upper bound for $\frac{1}{\{\omega_t^n\}} \frac{\omega_t^n}{\omega^n}$. Since $\chi \leq A\omega$ also by assumption, $PSH(X, \omega_t) \subset PSH(X, (A+1)\omega)$, and exponential boundedness implies the integral above is bounded by a constant independent of t and E' .

First, observe that if $\sup_X \psi_{E',\omega_t} \leq 1$, then

$$\{\omega_t^n\} = \int_{\bar{E}'} (\omega_t + dd^c\psi_{E',\omega_t})^n \leq \text{cap}_{\omega_t}(\bar{E}') \leq \text{cap}_{\omega_t}(E) \leq \text{cap}_{\omega_t}(X) = \{\omega_t^n\}.$$

This then implies that $\{\omega_t^n\}/\text{cap}_{\omega_t}(E) = 1$, and clearly then we are free to choose constants so that the desired inequality holds.

Next, if $\sup_X \psi_{E',\omega_t} > 1$, we can write, by the same logic

$$\left(\sup_X \psi_{E',\omega_t}\right)^{-n} = \left(\sup_X \psi_{E',\omega_t}\right)^{-n} \frac{\int_{\bar{E}'} (\omega_t + dd^c\psi_{E',\omega_t})^n}{\{\omega_t^n\}} \leq \frac{\int_{\bar{E}'} (\omega_t + dd^c(\psi_{E',\omega_t}/\sup_X \psi_{E',\omega_t}))^n}{\{\omega_t^n\}}.$$

This last term is bounded by $\{\omega_t^n\}^{-1} \text{cap}_{\omega_t}(E') \leq \{\omega_t^n\}^{-1} \text{cap}_{\omega_t}(E)$. So we obtain

$$\frac{1}{\{\omega_t^n\}} \int_{E'} \omega_t^n \leq \exp\left(-\delta \left(\frac{\{\omega_t^n\}}{\text{cap}_{\omega_t}(E)}\right)^{1/n}\right).$$

Taking a little care as E' approaches E , we are done. \blacksquare

4 References

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